SECOND COURSE IN ALGEBRA

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MATHEMATICAL TEXTS
FOR SCHOOLS

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SECOND COURSE IN ALGEBRA

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PREFACE

This revision of the "Second Course in Algebra" has been carried out in the same spirit as was the recent revision of the "First Course in Algebra" by the same authors. The exercises and problems, mainly new, have been carefully graded, and the exposition has been wholly rewritten. Some advantageous changes have been made in the order of topics, and some chapters for which no well-grounded demand exists have been omitted. Such simplifications of exercises and exposition have been made as are consistent with a standard course in third-semester algebra.

In the chapters devoted to a review of first-year algebra the fact was borne constantly in mind that the material would be handled by students who had not pursued the study of algebra during the preceding year. It was consequently thought desirable to have work in equations come much earlier than before. The subject of fractions, the topic usually most in need of review, has received full and careful treatment. Linear systems have been presented without the use of determinants. Instead of treating square root, radicals, and exponents in one chapter, the work under these topics has been made more accessible by giving a separate chapter to each. The needs of classes, even under almost identical conditions, differ widely, one class needing more review on a certain topic than does another. Consequently the review material has been expanded so as to afford ample work for any class. It is not intended, however, that all the exercises and problems should be solved by any one student.

The general plan of treatment of topics pertaining strictly to third-semester work is similar to that of the original edition, but the work on radical equations, on simultaneous equations involving quadratics, on logarithms, and on the binomial theorem has been considerably simplified.

In graphical work, the attention has been centered on the graphical representation of a function, on the graphical solution of an equation in one unknown and on that of a system in two unknowns. A few carefully selected statistical problems, each showing some striking feature when treated graphically, have been included. The whole is designed to secure with as little labor as possible the maximum of illumination and interest.

The authors are under obligations to many teachers from all parts of the country for helpful criticisms which have been of material assistance in planning and carrying forward this revision.

CONTENTS

Find the numerical value of:

11. $9x - 7$ if $x = 3$.

12. $2x^2 - 5x + 3$ if $x = 4$.

13. $t^3 - 3t^2 + 3t - 1$ if $t = 2$.

14. Does $4(2x - 5) + 15 = 3(x + 10)$ if $x = 7$?

15. Does $(t + 4)(t + 3) - (t + 1)(t + 2) = 42$ if $t = 8$?

16. Does $\frac{3}{x-1} - \frac{2x}{x+1} = \frac{3 + 5x - 2x^2}{x^2 - 1}$ if $x = 2$?

2. Similar terms. Terms which are alike in every respect except their coefficients are called **similar.**

3. Addition. In algebra, *addition* involves the uniting of similar terms which have the same or opposite signs into one term. For this we have the following rules:

I. To add two or more positive numbers, find the arithmetical sum of their absolute values and prefix to this sum the plus sign.

II. To add two or more negative numbers, find the arithmetical sum of their absolute values and prefix to this sum the minus sign.

III. To add a positive and a negative number, find the difference of their absolute values and prefix to this difference the sign of the number which has the greater absolute value.

Obviously $2 + 4 + 7 = 2 + 7 + 4 = 7 + 2 + 4$, etc. Even if we have a series of positive and negative numbers, the order in which they occur does not affect their sum. This principle of addition is called the *Commutative Law for Addition.*

For the addition of polynomials we have the

Rule. *Write similar terms in the same column.*

Find the algebraic sum of the terms in each column and write the results in succession with their proper signs.

4. Subtraction. For the subtraction of polynomials we have the

Rule. *Write the subtrahend under the minuend so that similar terms are in the same column.*

Then change mentally the sign of each term of the subtrahend and apply the rule for addition to each column.

EXERCISES

Add:

1. $12, -8, +4, -3$, and 6.
2. $4a, 3a, -7a, 6a$, and $-2a$.
3. $5a-3c+6, 2a-6c+11$, and $4a-c-9$.
4. $3s-4t+6, -7t+6s-8$, and $t+s+10$.
5. $x-2y^2-3z, 3y^2-4x+2z$, and $4z-y^2$.
6. $a^2-3a+1, -2a^2-7a+6$, and $3a^2-4+5a$.

Write so that x, y, or t shall have a polynomial coefficient:

7. $ax-2x$.

Solution. $ax-2x=(a-2)x$.

8. $ax+bx+cx$.
9. $5at-4bt-2t$.
10. $4t-3at-st$.
11. $at-a^2t-2st$.
12. $ay+by+y$.
13. $3ax-4bx+6x$.
14. $4x-abx-x$.
15. $7x-3ax-4a^2x$.

Write the following so that the binomial will have a binomial coefficient:

16. $(a-3)x-c(a-3)$.
17. $4(a-x)+c(-x+a)$.
18. $3(a+b)-c(a+b)$.
19. $6a(x-2c)-3(x-2c)$.
20. $3a(x-1)-2b(x-1)$.
21. $4b(3x-2)-8c(3x-2)$.
22. $4m(5a-3c)-6n(-3c+5a)$.

Subtract the second expression from the first:

23. $6a, 4a$.
24. $8a^3, 15a^3$.
25. $4x+3, 8x+6$.
26. $7x^2-10, 5x^2+20$.

27. $5x - 6,\ 2x + 8.$

28. $x^2 - 5x + 6,\ 2x^2 + 8x - 10.$

29. $a^2 - 4ac - 3c^2,\ 4a^2 + 10ac - c^2.$

30. $3a - 2b - 6c,\ 4a + 6b - 7c - 2.$

31. $3a^2 - 2c^2 - 6ac,\ 5a^2 + 4c^2 - 3ac.$

32. $a^3 - 3a^2c + 6ac^2,\ 7c^3 + 4a^2c - 2a^3.$

33. $x - 3y^2 + z - 4ac + 7ax,\ 4x - y^2 + 8 - 5ax + 9ac.$

34. $a^3 - c + 3x - a^2m - 8ac,\ 4a^3 + m - 8x - 10ac + 4a^2m.$

Find the expression which added to the first will give the second:

35. $3x^2 - 5x + 2,\ 6x^2 - 11x + 8.$

36. $4x^2 - 3cx + c^2,\ 10x^2 + 8cx - 9c^2.$

Find the expression which subtracted from the second will give the first:

37. $4a^2 - 2ab + b^2,\ 7a^2 - 10ab + 6b^2.$

38. $c^2 - 6cx + 8x^2,\ 9x^2 - 10cx + 4 + c^2.$

39. From the sum of $t^2 - 4t - 9$ and $3t^2 - 8t + 1$ take the sum of $3t - 6 + 4t^2$ and $5 - 8t^2 + 4t$.

40. From the sum of $ax - ac - 4c^2$ and $4c^2 - 3ac$ take the sum of $4c^2 - 8ax + a^2$ and $4ac + 3ax - 5c^2$.

ORAL EXERCISES

1. What name is given to each 3 in $a^3 + 3a$? Define both.

2. Distinguish between an exponent and a power. What is the meaning of 4 in x^4? of 2 in 3^2? of a in x^a?

3. What is the coefficient of x in $3a^2bx$? of a^2? of b?

4. What is the coefficient of x in the expressions $ax + x$? $4x - ax$? $cx - ax - x$? What is the meaning of 3 in $3x$? of 10 in $10x$? of a in ax?

5. What is meant by the absolute value of a number? Illustrate.

6. What is a literal exponent? Illustrate.

7. $x^3 \cdot x^5 = ?$ $x^a \cdot x^2 = ?$ $x^{a+2} \cdot x^3 = ?$

8. $x^6 \div x^2 = ?$ $x^a \div x^2 = ?$ $x^{2a+1} \div x^3 = ?$

9. How can the correctness of the result of addition be checked? of subtraction?

10. What is meant by arrangement of an algebraic expression with respect to a certain letter?

11. Arrange $a^4 + b^4 - 4a^3b - 6a^2b^2 + 4ab^3$ according to the descending powers of b.

12. Arrange $t^3 - 3t^4 - 5 + t - 2t^2$ according to the ascending powers of t; according to the descending powers of t.

13. Is arrangement of divisor and dividend in the same order desirable? Why?

5. Multiplication. In multiplying one term by another the sign of the product, the coefficient of the product, and the exponent of any letter in the product are obtained as follows:

I. The sign of the product is ***plus*** *if the multiplier and the multiplicand have like signs, and* ***minus*** *if they have unlike signs.*

II. The coefficient of the product is the product of the coefficients of the factors.

III. The exponent of each letter in the product is determined by the general law

$$n^a \times n^b = n^{a+b}.$$

For the multiplication of polynomials we have the

Rule. *Multiply the multiplicand by each term of the multiplier in turn, and add the partial products.*

27. $5x - 6,\ 2x + 8.$

28. $x^2 - 5x + 6,\ 2x^2 + 8x - 10.$

29. $a^2 - 4ac - 3c^2,\ 4a^2 + 10ac - c^2.$

30. $3a - 2b - 6c,\ 4a + 6b - 7c - 2.$

31. $3a^2 - 2c^2 - 6ac,\ 5a^2 + 4c^2 - 3ac.$

32. $a^3 - 3a^2c + 6ac^2,\ 7c^3 + 4a^2c - 2a^3.$

33. $x - 3y^2 + z - 4ac + 7ax,\ 4x - y^2 + 8 - 5ax + 9ac.$

34. $a^3 - c + 3x - a^2m - 8ac,\ 4a^3 + m - 8x - 10ac + 4a^2m.$

Find the expression which added to the first will give the second:

35. $3x^2 - 5x + 2,\ 6x^2 - 11x + 8.$

36. $4x^2 - 3cx + c^2,\ 10x^2 + 8cx - 9c^2.$

Find the expression which subtracted from the second will give the first:

37. $4a^2 - 2ab + b^2,\ 7a^2 - 10ab + 6b^2.$

38. $c^2 - 6cx + 8x^2,\ 9x^2 - 10cx + 4 + c^2.$

39. From the sum of $t^2 - 4t - 9$ and $3t^2 - 8t + 1$ take the sum of $3t - 6 + 4t^2$ and $5 - 8t^2 + 4t.$

40. From the sum of $ax - ac - 4c^2$ and $4c^2 - 3ac$ take the sum of $4c^2 - 8ax + a^2$ and $4ac + 3ax - 5c^2.$

ORAL EXERCISES

1. What name is given to each 3 in $a^3 + 3a$? Define both.

2. Distinguish between an exponent and a power. What is the meaning of 4 in x^4? of 2 in 3^2? of a in x^a?

3. What is the coefficient of x in $3a^2bx$? of a^2? of b?

4. What is the coefficient of x in the expressions $ax + x$? $4x - ax$? $cx - ax - x$? What is the meaning of 3 in $3x$? of 10 in $10x$? of a in ax?

6. Division. In dividing one term by another the sign of the quotient, the coefficient of the quotient, and the exponent of each letter in the quotient are obtained as follows:

I. The sign of the quotient is **plus** *when the dividend and the divisor have like signs, and* **minus** *when they have unlike signs.*

II. The coefficient of the quotient is obtained by dividing the coefficient of the dividend by that of the divisor.

III. The exponent of each letter in the quotient is determined by the law

$$n^a \div n^b = n^{a-b}.$$

The method of dividing one polynomial by another is stated in the

Rule. *Arrange the dividend and the divisor according to the descending powers of some common letter, called the letter of arrangement.*

Divide the first term of the dividend by the first term of the divisor and write the result for the first term of the quotient.

Multiply the entire divisor by the first term of the quotient, write the result under the dividend, and subtract, being careful to write the terms of the remainder in the same order as those of the divisor.

Divide the first term of the remainder by the first term of the divisor to get the second term of the quotient, and proceed as before until there is no remainder, or until the remainder is of lower degree in the letter of arrangement than the divisor.

EXERCISES

Perform the indicated division:

1. $(2x^2 - 5x + 3) \div (2x - 3)$.
2. $(6x^2 - 13x + 6) \div (3 - 2x)$.
3. $(3x^2 + 5xy - 2y^2) \div (x + 2y)$.
4. $(6a^2 + 23at - 55t^2) \div (3a - 5t)$.
5. $(6a^3 + 6a^2 - 28 - 26a) \div (2a + 4)$.

EXERCISES

Perform the indicated multiplication:

1. $(6x^2 - 3x + 4)3x$.
2. $(3x - 5)(4x + 3)$.
3. $(2s - 3t)(4s + 5t)$.
4. $(x^2 - x + 2)(3x - 4)$.
5. $(3x^2 - 2x + 6)(7 - 3x^2 - x)$.
6. $(2x^2 - 5x + 3)(x^2 - 5x + 6)$.
7. $(a^4 + 2a^2 - 4)(a^2 - 2a - 3)$.
8. $(t^3 - 2t + 6)(t^4 - 3t^3 - t^2)$.
9. $(2s^2 - 3st + t^2)(s^2 + 5st - 4t^2)$.
10. $(t^4 + 8 + 4t^2)(t^4 + 8 - 4t^2)$.
11. $(a^2 - ac + c^2)(a^2 + c^2 + ac)$.
12. $(a^2 + 2ab + b^2)(a^2 + b - ab)$.
13. $(t^3 - t^2 + t)(at^2 + a + at)$.
14. $(4h^2 + 6k^2 + 9hk)(4h^2 + 6k^2 - 9hk)$.
15. $(3at^2 - 2at^3 + 5a)(6at^4 - 2at - 4at^2)$.
16. $(2a^2 - 3c^6 + 4ac^3)(2a^2 - 3c^6 - 4ac^3)$.
17. $\left(2 + \frac{3s}{4} - \frac{2s^2}{3}\right)\left(1 - \frac{4s}{5} + \frac{s^2}{6}\right)$.
18. $\left(\frac{a}{2} - \frac{2a^2}{3} - \frac{a^3}{4}\right)\left(\frac{a}{2} - \frac{2a^2}{3} - \frac{a^3}{4}\right)$.
19. $(t^2 + 1 + t)(t^4 + 1 - t^2)(t^2 + 1 - t)$.
20. $(x^2 + 9y^2 + 16 + 3xy - 4x + 12y)(x - 3y + 4)$.

Find the value of:

21. $3x^2 + 2x + 5$ if $x = 5$.
22. $3t^2 - 4t + 8$ if $t = -3$.
23. $9 - 8t + 5t^2 - 3t^3$ if $t = 2$.
24. $2a^3 - 3a^2c + 4ac^2 - 3c^2$ if $a = 3$ and $c = -2$.
25. Does $15(x - a) - 6(x + a) = 3(5a - 3x)$ if $x = 2a$?
26. Does $ax(a + 3) + a(10 - a^2) = x + 3$ if $x = a - 3$?
27. Does $\frac{3}{x-1} - \frac{2x}{x+1} = \frac{3 + 5x - 2x^2}{x^2 - 1}$ if $x = 0$? if $x = -2$?

When one parenthesis incloses another, either the outer or the inner parenthesis may be removed first. Usually it is best to use the

Rule. *Rewrite the expression, omitting the innermost parenthesis, changing the signs of the terms which it inclosed if the sign preceding it be minus and leaving them unchanged if it be plus.*

Combine like terms that may occur within the new innermost parenthesis.

Repeat these processes until all the parentheses are removed.

EXERCISES

Remove parentheses and simplify:

1. $a+(a-b)-(a-3b)$.

2. $a-(a-c)+(2c-3a)$.

3. $3b-[(a-b)-(c-a)]$.

4. $2c-2(a-c)+3(c-a)$.

5. $x-y+3(x-y)-4(2x-y)$.

6. $4x-a+[-(3c-x)-(2a-3x)]$.

7. $a-[-(a-3)+(3c-2a)-5a]+6c$.

8. $x-3-(a-2x)-[2(a-x-5)-3(6-2a)]$.

9. $2t^2-3t-2t(3+t)$.

10. $x^2-3-(x-1)(x-2)$.

11. $4x^2-3a^2-(a-2x)(3x-a)$.

12. $6x+(3c-8x+2)-(c-x-2)$.

13. $6x-[-(a-c)+(3c-4a)]$.

14. $7c-[(3c-4)-6-(4x-3a-c)]$.

15. $4x-2(x-3)-3[x-3(4-2x)+8]$.

16. $6x-4(3-5x)-4[2(x-4)+3(2x-1)-(x-7)]$.

17. $3x-2[1-3(2x-3-a)-5\{a-(3x-2a)-4\}]$.

18. $2t^2-7t-(2t-1)(t+1)$.

19. $(x-4)(x-3)-(x-3)(x+2)$.

20. $(a+b)a-(a-b)b+3ab$.

21. $3a(a-b)-(a+b)(a-b)$.

22. $(x-3)(x-4)-(x-5)(x+3)$.

8. Important special products. Certain products are of frequent occurrence. These forms should be memorized so that one can write or state the result without the labor of actual multiplication.

I. For the square of the sum of two terms we have the formula

$$(a+b)^2 = a^2 + 2ab + b^2.$$

This expressed verbally is:

The square of the sum of two terms is the square of the first term plus twice the product of the two terms plus the square of the second term.

II. For the square of the difference of two terms we have the formula

$$(a-b)^2 = a^2 - 2ab + b^2.$$

This expressed verbally is:

The square of the difference of two terms is the square of the first term, minus twice the product of the two terms, plus the square of the second term.

III. For the product of the sum and the difference of two terms we have the formula

$$(a+b)(a-b) = a^2 - b^2.$$

This expressed verbally is:

The product of the sum and the difference of two terms equals the difference of their squares taken in the same order as the difference of the terms.

IV. For the product of two binomials having a common term we have the formula

$$(x+a)(x+b) = x^2 + (a+b)x + ab.$$

This expressed verbally is:

The product of two binomials having a common term equals the square of the common term, plus the algebraic sum of the unlike terms multiplied by the common term, plus the algebraic product of the unlike terms.

V. The square of the polynomial $a + b - c$ gives the formula

$$(a + b - c)^2 = a^2 + b^2 + c^2 + 2ab - 2ac - 2bc.$$

This expressed verbally is:

The square of any polynomial is equal to the sum of the squares of each of the terms plus twice the algebraic product of each term by every term that follows it in the polynomial.

VI. The cube of the binomial $a + b$ gives the formula

$$(a + b)^3 = a^3 + 3a^2b + 3ab^2 + b^3.$$

This expressed verbally is:

The cube of the sum of two numbers equals the cube of the first, plus three times the square of the first times the second, plus three times the first times the square of the second, plus the cube of the second.

VII. Similarly,

$$(a - b)^3 = a^3 - 3a^2b + 3ab^2 - b^3.$$

This can be expressed verbally in a manner similar to VI.

ORAL EXERCISES

State the result of the indicated multiplication:

1. $(x + 5)^2$.	**6.** $(2x - 1)^2$.	**11.** $(x^2 - x)^2$.
2. $(x + 7)^2$.	**7.** $(3x - 5)^2$.	**12.** $(a^2 + 3a)^2$.
3. $(2x + 1)^2$.	**8.** $(ax - 7)^2$.	**13.** $(x - a)(x + a)$.
4. $(3x + 2)^2$.	**9.** $(x + 7a)^2$.	**14.** $(x - 5)(x + 5)$.
5. $(x - 3)^2$.	**10.** $(2x - 3y)^2$.	**15.** $(2x - 1)(2x + 1)$.

16. $(ax + 3)(ax - 3)$.

17. $(4x - 3c)(4x + 3c)$.

18. $(ax - c)(ax + c)$.

19. $(4a + c)(c - 4a)$.

20. $(3a + 2x)(2x - 3a)$.

21. $(x + 3)(x + 4)$.

22. $(x + 2)(x + 5)$.

23. $(x + 1)(x + 7)$.

24. $(x - 3)(x - 5)$.

25. $(x - 2)(x - 7)$.

26. $(x - 3)(x + 4)$.

27. $(x + 6)(x - 7)$.

28. $(x + 2)(x - 9)$.

29. $(x - 5)(x + 10)$.

30. $(ax - 3)(ax + 5)$.

31. $(2x - 1)(2x + 3)$.

32. $(3x + 1)(3x + 4)$.

33. $(4x - 2)(4x + 3)$.

34. $(a^2 - 3a)(a^2 + 4a)$.

35. $(a + b + c)^2$.

36. $(a + c + x)^2$.

37. $(a + c - x)^2$.

38. $(a - c + x)^2$.

39. $(a + c + 1)^2$.

40. $(a - c + 1)^2$.

41. $(a + c + 2)^2$.

42. $(a + c + 6)^2$.

43. $(2a + c + 1)^2$.

44. $(a + 2x - 3)^2$.

45. $(a - 3x + 2)^2$.

46. $(a - 3b - c)^2$.

47. $(a + b + c + d)^2 =$
$$a^2 + b^2 + c^2 + d^2 + 2ab + 2ac + 2ad + 2bc + 2bd + 2cd.$$

48. $(a + c + x + d)^2$.

49. $(a + b + c + x)^2$.

50. $(a + b + c - x)^2$.

51. $(a + b - c - x)^2$.

52. $(x - y - c - a)^2$.

53. $(x + 2a + c + y)^2$.

54. $(3x + a + y - c)^2$.

55. $(a + 2b + 3c + d)^2$.

56. $(a - 2c + x - 3y)^2$.

57. $(2a - c - 3x + y)^2$.

58. $(a + c)^3$.

59. $(a + 2)^3$.

60. $(a + 3)^3$.

61. $(a - x)^3$.

62. $(a - 2)^3$.

63. $(a + 1)^3$.

64. $(1 - a)^3$.

65. $(a - 3)^3$.

66. $(a + 2x)^3$.

67. $(3a - x)^3$.

68. $(3a + 2x)^3$.

69. $(2a - 3x)^3$.

70. $(a^2 - a)^3$.

71. $(x^2 + 2x)^3$.

72. $(2a^2 - a)^3$.

EXERCISES

Find the following products and expand the results:

1. $[(x + y) + 1][(x + y) - 1]$.
2. $[(x + a) + 3][(x + a) - 3]$.
3. $[(x - a) + 3][(x - a) - 3]$.
4. $[(x + 4) + c][(x + 4) - c]$.
5. $[(2a - b) + c][(2a - b) - c]$.
6. $[x + (b + c)][x - (b + c)]$.
7. $[x + (b - c)][x - (b - c)]$.
8. $[3 + (x - y)][3 - (x - y)]$.
9. $[10 - (a - 5)][10 + (a - 5)]$.
10. $[4x + (2y - x)][4x - (2y - x)]$.

11. From each corner of a square piece of tin of side a inches a square of side b inches is cut. By turning up the sides an open box is formed. Show that $a^2 - 4b^2$ is the area of the inside of the box in square inches.

12. Express the area $a^2 - 4b^2$ of Exercise 11 as the product of two binomials.

13. Using the results of Exercise 12, find by a short method the area of the inside of the box if $a = 12$ and $b = 3$; if $a = 95$ and $b = 5$.

14. The dimensions of a rectangular box are d, $d + 3$, and $d + 6$. Express (a) the sum of the edges, (b) the total outer surface, and (c) the volume of the box.

15. Assume each dimension of the box of Exercise 14 equal to n inches and solve as before.

16. If two equal boxes of dimensions n, $n - 4$, and $n + 5$ are placed end to end, find (a) the sum of the outer edges, (b) the outer surface, and (c) the combined volume.

17. The area of a circle is given by the formula πr^2, in which $\pi = \frac{22}{7}$ and r equals the radius of the circle. What is

[redacted]

the area of a circular ring left by cutting a circle of radius r from the center of a circle of radius R?

18. The surface of a cylinder is $2\pi r(h+r)$, in which r is the radius of the base and h is the height of the cylinder. Find the amount of tin in 1250 cylindrical cans which have a circular base of radius 2 inches and a height of 5 inches.

19. How much less tin is used in making a cylindrical can of height h inches and radius r inches than in making one of height $h+2$ inches and radius $r+2$ inches?

20. Would more tin be used in constructing a cylindrical can of height h and radius $r+2$ or of height $h+2$ and radius r? How much more?

21. Find the total surface of a cylinder the radius of whose base is $r+8$ and whose height is $r-5$.

22. The formula for the volume of a cylinder is $\pi r^2 h$, in which r equals the radius of the circular base and h the height. Find the number of gallons contained in the 1250 cans of Exercise 18, given that 1 gallon contains 231 cubic inches.

23. Find the volume of a cylinder the radius of whose base is $r-5$ and whose height is $r+5$.

24. The formula for the surface of a sphere is $4\pi r^2$, in which r is the radius of the sphere. Compare the sum of the surfaces of 1728 balls of radius one-half inch with the surface of a ball of radius 9 inches.

25. Compare the surface of a sphere of radius $3n$ with the surface of a sphere of radius n.

26. The formula for the volume of a sphere is $\frac{4}{3}\pi r^3$. How many spherical balls of radius 2 inches can be made from a spherical ball of lead of radius 12 inches?

27. Write the formula for the volume of a sphere whose radius is $r-5$ inches.

28. Compare the volume of two spheres of radii r and $2r$ respectively.

CHAPTER II

LINEAR EQUATIONS IN ONE UNKNOWN

9. Definition of an equation. An **equation** is a statement of the equality between two equal numbers or number symbols.

Thus $a(a-2) = a^2 - 2a$ and $x + 5 = 7$ are equations.

Equations are of two kinds — *identities* and *equations of condition.*

10. Identities. An arithmetical or an algebraic **identity** is an equation in which either the two members are alike, term for term, or become so if indicated operations be performed.

Thus $15 - 8 = 6 + 1$ and $(a+b)(a-b) = a^2 - b^2$ are identities, for in each, if the indicated operations be performed, the two members become precisely alike.

An identity involving letters is true for *any* set of numerical values of the letters in it.

Thus the identity $a(b-c) = ab - ac$ becomes $2(9-5) = 18 - 10$, or $8 = 8$, when, for example, $a = 2$, $b = 9$, and $c = 5$.

11. Equation of condition. An equation which is true only for certain values of a letter in it, or for certain sets of related values of two or more of its letters, is an **equation of condition,** or simply an equation.

Thus $2x + 5 = 17$ is true for $x = 6$ only; and $x + 2y = 10$ is true for $x = 8$ and $y = 1$ and for many other pairs of values for x and y, but it is not true for $x = 5$ and $y = 2$ and for many other (though not for all) pairs of values for x and y.

12. Satisfying an equation. A number or literal expression which, being substituted for the unknown letter in an equation, changes it to an identity, is said to **satisfy** the equation.

Thus $x = 6a$ satisfies the equation $2x + 5a = 17a$, for on substituting $6a$ for x we have $2 \cdot 6a + 5a = 17a$, or $17a = 17a$, which is an identity.

After the substitution is made it is usually necessary to simplify each member before the identity becomes apparent.

13. Root of an equation. A **root of an equation** is any number or number symbol which satisfies the equation.

Thus 8 is the root of the equation $3x + 2 = 26$, for it satisfies the equation.

14. Axioms. An **axiom** is a statement the truth of which is accepted without proof. Some of the axioms most frequently used are

Axiom I. *If the same number is added to each member of an equation, the result is an equation.*

Axiom II. *If the same number is subtracted from each member of an equation, the result is an equation.*

Axiom III. *If each member of an equation is multiplied by the same number, the result is an equation.*

Axiom IV. *If each member of an equation is divided by the same number (not zero), the result is an equation.*

Each of the foregoing axioms is used in the solution of the

EXAMPLE

Solve $\frac{5x}{2} - \frac{1}{2} = x + 7.$

Solution. $$\frac{5x}{2} - \frac{1}{2} = x + 7. \qquad (1)$$

Multiplying (1) by 2, (Ax. III)

$$5x - 1 = 2x + 14. \qquad (2)$$

Adding 1 to each member of (2), (Ax. I)

$$5x = 2x + 15. \qquad (3)$$

Subtracting $2x$ from each member of (3), (Ax. II)

$$3x = 15. \qquad (4)$$

Dividing (4) by 3, (Ax. IV)

$$x = 5. \qquad (5)$$

Check. Substituting 5 for x in (1), we have

$$\tfrac{25}{2} - \tfrac{1}{2} = 5 + 7, \text{ or } 12 = 12.$$

Since substituting 5 for x satisfies (1), 5 is the root of (1).

15. Transposition. In the solution of the foregoing section by the application of Axiom I to (2), the term -1 is omitted from the first member and $+1$ is combined with the second member. Again, by applying Axiom II to (3), the term $+2x$ is omitted from the second member and $-2x$ is combined with the first member.

It thus appears that a term may be omitted from one member of an equation, provided the same term with its sign changed from + to − or from − to + is written in or combined with the other member. This process is called **transposition.**

Hereafter, in order to simplify an equation, instead of subtracting a number from each member or adding a number to each member, as illustrated in the foregoing example, the student should use transposition, since it is usually more rapid and convenient. He should, however, always remember that the *transposition of a term is really the subtraction of that term from each member of the equation.*

16. Equivalent equations. Two or more equations in one unknown, even if of very different form, are **equivalent** if all are satisfied by every value of the unknown which satisfies any one of them.

Equations (2), (3), (4), and (5) of section 14 are each equivalent to equation (1) and to each other, for all are satisfied by the same value of the unknown.

Of the four axioms or assumptions of section 14 we shall make constant use. If the "same number" referred to in each is expressed arithmetically, the result is always an equation *equivalent* to the *original* one. Further, *if identical expressions involving the unknown* be added to or subtracted from each member of an equation, the resulting equation is equivalent to the first. If, however, both members of an equation be multiplied by identical expressions containing the unknown, the resulting equation *may not* be equivalent to the original one.

Multiplying each member of the equation $x - 2 = 3$ by $x - 1$, we get $x^2 - 3x + 2 = 3x - 3$, or $x^2 - 6x + 5 = 0$. Now this last equation has the roots 1 and 5, whereas the given equation has the root 5 only. Here the root 1 was introduced by multiplying the given equation by $x - 1$. Results obtained from the use of Axiom III with multipliers which contain an unknown should always be carefully checked. When a root is obtained which does not satisfy the original equation, this root should be rejected.

The use of Axiom IV when the divisor contains the unknown may result in the loss of a root which the process of checking will not discover. *If an equation is divided by a factor containing an unknown*, this factor should be set equal to zero. The root thus obtained is a root of the given equation.

For example, if each member of $x^2 - 4 = 3x + 6$ is divided by $x + 2$, the result is $x - 2 = 3$, whence $x = 5$. But $x = -2$ satisfies $x^2 - 4 = 3x + 6$. This root was lost by dividing by $x + 2$.

With these and with certain other rare exceptions which will be noted later, the application of the axioms will produce an equation equivalent to the given one.

For solving equations in one unknown which do not involve fractions we have the

Rule. *Free the equation of any parentheses it may contain.*
Transpose and solve for the unknown involved.
Reject all values for the unknown which do not satisfy the original equation.

Checking the solution of an equation is often called testing or **verifying** the result. For this we have the

Rule. *Substitute the value of the unknown obtained from the solution in place of the letter which represents the unknown in the original equation. Then simplify each member of the resulting identity until the two members are seen to be identical.*

If the correct substitution of the root for the unknown does not transform the equation into an identity, an error has been made in the solution.

EXERCISES

Solve, and check the results as directed by the teacher:

1. $5x + 1 = 2x + 7.$
2. $6x + 10 = 10x + 2.$
3. $1 + x + 5x + 17 = 0.$
4. $15x = 3(4x - 5).$
5. $2(x + 2) - (x + 5) = 0.$
6. $3(2x - 1) - (5x - 1) = 0.$
7. $3(2x - 7) - 2(5 - 2x) + 1 = 0.$
8. $3(x + 2) - 5(2x - 3) = 0.$
9. $6(x + 4) - 4(x + 2) = 0.$
10. $4(6t + 2) - 3(7t + 3) = 0.$
11. $6(4t - 5) - 11(2t - 3) = 0.$
12. $6(2x - 1) + 4(3 - 4x) = 0.$
13. $4(4x - 1) + 3 - 2(3 + x) = 0.$
14. $5n - 9(2n + 4) - 2(n - 9) = 0.$
15. $4(x - 2) + 3(2 - x) - 3x = 6(x + 1).$

16. $3n - 5(4 - n) = 5 - 3(1 + n)$.

17. $(x + 1)(x - 2) = x^2 + 3$.

18. $(x + 5)(x + 1) = (x - 3)(x - 2) + 10$.

19. $(4x - 3)(2x - 5) - (4x - 7)(2x - 1) = 0$.

20. $(x + 2)^2 + 48 = (x - 4)^2$.

21. $(x + 3)^2 + 40 = (x + 5)^2$.

22. $(2x + 3)^2 = 4(1 - x)^2$.

23. $(x + 4)^2 - (2 - x)^2 = 84$.

24. $(x + 3)(6x + 5) - (2x + 4)(3x - 8) = 38$.

25. $(1 - x)(x + 2) + (x + 3)(x + 4) = 0$.

26. $(y - 4)(6 - y) + (y + 2)(y - 4) = 0$.

27. $(x + 4)(x + 3) = (x + 2)(x + 1) + 42$.

28. $(2v - 3)(3v + 2) - (4 - 6v)(1 - v) = 0$.

29. $(5x - 3)(4 - 6x) + (3x + 4)(10x - 21) - 9 = 0$.

30. $3(x + 2)(x - 4) + 5(x - 1)(x + 3) = 4(2x - 1)(x - 2) + 1$.

31. $x - 2a = 4a - x$.

32. $c - x = x - c$.

33. $8s - x = x - 4r$.

34. $ax - 2ab = 4ab - ax$.

35. $3ax - 5ac = 3ac - ax$.

36. $3c(x - 2a) = 2c(a - x)$.

37. $m(x - 5a) - 3m(a - x) = 0$.

38. $x(b + a) = ab + a^2$.

39. $x(a - c) = a^2 - c^2$.

40. $ax - 2cx = a^2 - 4c^2$.

41. $2ax - 4a^2 + 4a = 1 + x$.

42. $ax - a^2 + 5a = 6 + 3x$.

43. $ax + 1 - a^3 = x$.

44. $2mx + 5m - 3 = 3x + 2m^2$.

45. $(x + a)(x + b) = x^2 + 2a^2 + 3ab$.

46. $(a - c)(x - m) = (m - c)(x - a)$.

47. $7ax - 31ab = 14a^2 + 15b^2 - 5bx$.

48. $a^2x - a^3 + 3ax = 3 - 10a + x$.

49. $c(1 + x) + m(x + 1) - x(m + c + 1) = 0$.

50. $m(m - 2x) + 2am = a(2x - a)$.

17. Solution of problems. In the solution of problems leading to simple equations the following steps are necessary:

I. Read the problem carefully and find the facts which will later be expressed by the equation.

II. Represent the unknown number by a letter and express any other unknown involved in terms of this letter.

III. Express the conditions stated in the problem as an equation involving this letter.

IV. Solve the equation.

V. Check by substituting in the problem the value found for the unknown.

In the preceding sentence the words "in the problem" are of importance, for substituting the value found in the equation would not detect any errors made in translating the words of the problem into the equation.

PROBLEMS

1. The sum of two consecutive numbers is 975. Find the numbers.

2. One number is five times another, and their sum is 102. What are the numbers?

3. The sum of two numbers is 54. Twice one of them equals ten times the other. What are the numbers?

4. The sum of three consecutive even numbers is 1044. Find the numbers.

5. The product of two consecutive even numbers is 1416 less than the product of the next two consecutive even numbers. Find the numbers.

6. Of four consecutive numbers the product of the second and fourth exceeds the product of the first and third by 201. Find the numbers.

7. One pupil is four years older than another. Eight years ago the first was twice as old as the second. Find their ages now.

8. Two men are 48 and 18 years of age respectively. How many years hence will the older be twice as old as the younger?

9. One man is three times as old as another. Fifteen years ago the first was six times as old as the second. Find their ages now.

10. A's age is double B's. Twelve years ago B's age was one fourth of A's. How old is each?

11. A is twice as old as B, and C is three times as old as D. B is 8 years older than D. In ten years the sum of their ages will be 113. How old is each?

12. If each side of a square is increased 7 feet, its area will be increased 329 square feet. Find the side of the square.

13. A certain rectangle is 8 feet longer than it is broad. If it were 2 feet shorter and 5 feet broader, its area would be 60 square feet greater. What are its length and breadth?

14. A certain rectangular plot of ground is 15 yards longer than it is wide. If it were 20 yards shorter, it would have to be 40 yards wider in order to have the same area. What are its dimensions?

15. A certain square plot is surrounded by a border 6 feet wide. The area of this border is 816 square feet. What is the side of the square?

16. A certain picture is 4 inches longer than it is wide, and the frame is 2 inches wide. The area of the framed picture is 192 square inches greater than that of the picture alone. What are the dimensions of the picture?

17. Can a rectangle of perimeter 112 inches be drawn which has a length 5 inches greater than twice the width? If so, give its dimensions.

18. A sum of $15.75 consists of dollars, quarters, and dimes. If there are 6 more quarters than dollars, and twice as many dimes as quarters, find the number of coins of each kind.

19. A certain sum consisting of quarters, dimes, and pennies amounts to $8.62. The number of dimes equals twice the number of quarters, while the number of dimes and quarters together is 2 greater than the number of pennies. Find the number of coins of each kind.

20. A collection of 109 coins is made up of quarters, dimes, and nickels. There are 7 fewer dimes than quarters, and 3 less than five times as many nickels as dimes. Find the amount of the collection.

21. The sum of the digits of a certain two-digit number is 14. If the order of the digits is reversed, the number is decreased by 36. Find the number.

22. Change the word "decreased" to "increased" in Problem 21 and solve.

23. The digits of a certain three-digit number are consecutive odd numbers. If the sum of the digits is 15, find the number.

Facts from Geometry. The area of a circle is the square of the radius multiplied by π ($\pi = \frac{22}{7}$ approximately). This is expressed by the formula $A = \pi R^2$.

The circumference of a circle equals the diameter times π. The usual formula is $C = 2\pi R$.

24. If the radius of a given circle is increased 14 inches, the area is increased 1232 square inches. Find the first radius.

25. By adding 2 inches to the radius of a circle whose radius is 8 inches, how much is the circumference increased? the area?

26. Substitute R inches for 8 inches in Problem 25 and solve. Interpret your results.

27. Imagine that a circular hoop 1 foot longer than the circumference of the earth is placed about the earth so that it is everywhere equidistant from the equator and lies in its plane. How far from the equator will the hoop be?

28. Compare the result of Problem 27 with the one obtained when a similar process is carried out with a sphere 18 inches in diameter, instead of with the earth.

29. If the height of a square is increased 4 feet and its length is increased twice that amount, the area of the figure will be increased 248 square feet. Find the side of the square and the area of the rectangle.

30. The rope attached to the top of a flagpole is 5 feet longer than the pole. The lower end of the rope just reaches the ground when taken to a point 25 feet from the base of the pole. Find the height of the pole.

31. The length of a given rectangle is three times its width. A second rectangle is 9 inches shorter and 1 inch wider than the first, and has a perimeter one half as great. Find the dimensions of each rectangle.

CHAPTER III

FACTORING

18. Definition of factoring. **Factoring** is the process of finding the two or more algebraic expressions whose product is equal to a given expression.

In multiplication we have two factors given and are required to find their product. In division we have the product and one factor given and are required to find the other factor. In factoring, however, the problem is a little more difficult, for we have only the product given, and our experience in multiplication and division is called upon to enable us to determine the factors.

19. Rational expressions. A **rational** algebraic expression is one which can be written without the use of indicated roots of the letters involved.

Thus 2, $5x$, $3y-\sqrt{2}$, and a^2 are rational expressions. In this chapter factors which involve radicals will not be sought.

20. Integral expressions. If a rational expression can be written so as not to involve an indicated division in which an unknown letter occurs in a denominator, it is said to be **integral**.

Thus 3, $7a$, $\frac{a}{5}$, and $4x-3$ are integral expressions. In this chapter factors which involve fractions will not be sought.

21. Prime factors. An integral expression is **prime** when it is the product of no two rational integral expressions except itself and 1.

It must be remembered that to factor an integral expression means to resolve it into its prime factors.

The methods of this chapter enable one to factor integral rational expressions in one letter which are not prime, as well as some of the simpler expressions in two letters. No attempt is made even to define what is meant by prime factors of expressions which are not rational and integral.

There is no simple operation the performance of which makes us sure that we have found the *prime* factors of a given expression. Only insight and experience enable us to find prime factors with certainty.

A partial check that may be applied to all the exercises in factoring consists in actually multiplying together the factors that have been found. If the result is the original expression, correct factors have been found, though they may not be prime factors.

22. Polynomials with a common monomial factor. The type form is

$$ab + ac - ad.$$

Factoring, $ab + ac - ad = a(b + c - d)$.

ORAL EXERCISES

Factor:

1. $3a + 6$.
2. $5x + 15$.
3. $a^2 + a$.
4. $2c - 6c^2$.
5. $9x^2 - 3x$.
6. $cd + c^2d^2$.
7. $ax^2 - a^2x$.
8. $4cx - 8c^2$.
9. $14h - 21h^2k$.
10. $5ax - 2ax^3$.
11. $2c + 4c^2 - 2cd$.
12. $4a - 10a^2 - 2a^3$.
13. $6ac - 3bc + 3c$.
14. $10x + 15x^3 - 5xy$.
15. $14a^4 - 7a^3 + 7a^2 - 7a$.
16. $3c^2 - 6c^4 + 9c^3 - 15c^4$.
17. $6r^2s^2 - 3r^3s^2 + 3r^2s^3 - 3r^3s^3$.
18. $10x^3y^3 - 2xy^3 + 2x^2y^3 - 2x^4y^4$.

23. Polynomials which may be factored by grouping terms and taking out a common binomial factor. The type form is

$$\boldsymbol{ax + ay + bx + by.}$$

Factoring,

$$\begin{aligned} ax + ay + bx + by &= (ax + ay) + (bx + by) \\ &= a(x + y) + b(x + y) \\ &= (x + y)(a + b). \end{aligned}$$

EXERCISES

Separate into polynomial factors:

1. $2(a + b) + x(a + b)$.

2. $3(b + 5) + a(b + 5)$.

3. $5x(a - c) + y(a - c)$.

4. $2a(x - 2y) + b(x - 2y)$.

5. $a(c - d) - b(c - d)$.

6. $2(x - y) - x(x - y)$.

7. $h(m + 3n) - 2k(m + 3n)$.

8. $2r(5x - 4y) - 9s(5x - 4y)$.

9. $-2a(3h - k) - b(3h - k)$.

10. $x(r - s) + y(s - r)$.

Hint. Write in the form $x(r - s) - y(r - s)$.

11. $2a(c - 3d) + b(3d - c)$.

12. $5r(3m - 2n) - 2s(2n - 3m)$.

13. $ac + ad + bc + bd$.

14. $ac + 2cx + 3ay + 6xy$.

15. $mx - my + nx - ny$.

16. $cd - 3cf + 2d - 6f$.

17. $ahr + akr - ahs - aks$.

18. $r^2s + 2rs - 3r^2t - 6rt$.

19. $4hm + 8hn - 6km - 12kn$.

20. $2a^2 - 2ax - ac + cx$.

21. $x^3 - 3xy^2 - 3x^2y + 9y^3$.

22. $ax + ay + bx + by + cx + cy$.

23. $mr - 2r + ms - 2s + mt - 2t$.

24. Trinomials which are perfect squares. The type form is

$$a^2 \pm 2\,ab + b^2.$$

Factoring, $\quad a^2 \pm 2\,ab + b^2 = (a \pm b)^2.$

ORAL EXERCISES

Separate into binomial factors:

1. $a^2 + 2\,ax + x^2.$
2. $x^2 - 2\,xy + y^2.$
3. $m^2 - 2\,mn + n^2.$
4. $x^2 + 4\,x + 4.$
5. $4 - 4\,x + x^2.$
6. $a^2 - 4\,ab + 4\,b^2.$
7. $r^2 - 10\,rs + 25\,s^2.$
8. $9 + 6\,a + a^2.$
9. $16 - 8\,ax + a^2x^2.$
10. $9\,x^2 - 12\,xy + 4\,y^2.$
11. $1 + 10\,ab + 25\,a^2b^2.$
12. $(a+b)^2 - 2(a+b) + 1.$
13. $(r-s)^2 - 6(r-s) + 9.$
14. $4(a+2)^2 - 12\,c(a+2) + 9\,c^2.$
15. $9 + 6(a+x) + (a+x)^2.$
16. $16 - 8(a-2\,x) + (a-2\,x)^2.$
17. $(a+b)^2 + 4(a+b)(c+2) + 4(c+2)^2.$
18. $(a+b)^2 - 6(a+b)(c-d) + 9(c-d)^2.$
19. $a^{2n} - 12\,a^n + 36.$
20. $x^{2a} - 14\,x^ay^b + 49\,y^{2b}.$

25. A binomial the difference of two squares. The type form is

$$a^2 - b^2.$$

Factoring, $\quad a^2 - b^2 = (a+b)(a-b).$

More generally,

$$\begin{aligned} a^2 + 2\,ab + b^2 - c^2 + 2\,cd - d^2 \\ &= a^2 + 2\,ab + b^2 - (c^2 - 2\,cd + d^2) \\ &= (a+b)^2 - (c-d)^2 \\ &= (a+b+c-d)(a+b-c+d). \end{aligned}$$

ORAL EXERCISES

Factor:

1. $a^2 - x^2$.
2. $m^2 - n^2$.
3. $a^2 - 4$.
4. $x^2 - 9$.
5. $16 - x^2$.
6. $a^2 - 16b^2$.
7. $1 - 25c^2$.
8. $9x^2y^2 - 1$.
9. $16x^2 - 25y^2$.
10. $25a^2 - 36b^2c^2$.
11. $36a^2b^2 - 49c^2d^2$.
12. $a^4 - 25b^4$.
13. $x^4y^4 - 9$.
14. $x^2y^8 - 64z^4$.
15. $81a^2 - 100b^4c^6$.
16. $a^4 - 16$.

HINT. Find three factors.

17. $a^4 - b^4$.
18. $x^4 - 1$.
19. $x^8 - 1$.
20. $x^4 - 81$.
21. $16 - a^4$.
22. $625 - x^4$.
23. $81 - c^4$.
24. $x^{2m} - y^{2n}$.
25. $a^{2m} - b^{2n}$.
26. $c^{4a} - d^{4b}$.

EXERCISES

Factor:

1. $a^4 - x^8$.
2. $x^4y^4 - z^4$.
3. $(a-2)^2 - c^2$.
4. $4(x+3)^2 - y^2$.
5. $16(x-y)^2 - z^4$.
6. $9(x-3y)^2 - 16z^4$.
7. $a^2 - (b+c)^2$.
8. $4r^2s^2 - (r-s)^4$.
9. $9m^2 - 4(n+3)^4$.
10. $(a-c)^2 - (d+e)^2$.
11. $a(r+2s)^2 - a(x-y)^2$.
12. $x^2(2h-k)^2 - x^2(m-2n)^4$
13. $4a(a-x)^2 - 9a(c-2d)^4$.
14. $a^2x(x-y)^2 - b^2x(3c-d)^2$.
15. $m^2 + 2mn + n^2 - (x^2 - 2xy + y^2)$.
16. $x^2 - 2xy + y^2 - a^2 - 2ab - b^2$.
17. $x^2 + 6x + 9 - y^2 + 2yz - z^2$.
18. $a^2 - 22a + 121 - b^2 + 20bc - 100c^2$.
19. $2 - 8x^2 + 8x^4 - 2y^2 - 8yz - 8z^2$.
20. $a^2b^2 + 2ab + 1 - c^2 + 10cd - 25d^2$.
21. $4c^2 - a^2 - 2ab - b^2$.
22. $50a^2b^2 - 8b^2 + 8bc - 2c^2$.

23. $49x^4 - 49y^2 - 14y - 1.$

24. $121x^8 - 1 - 18y - 81y^2.$

25. $a^2 - b^2 - (a - b).$

26. $x^2 - y^2 + (ax + ay).$

27. $r^2 - rs - (r^2 - s^2).$

28. $r^2 + rs - (r^2 - s^2).$

29. $m^2 - n^2 - m - n.$

30. $m + n - m^2 + n^2.$

31. $x^2 - 4y^2 + x - 2y.$

32. $r^2 - r - 3s - 9s^2.$

26. The quadratic trinomial. The type form is

$$x^2 + bx + c.$$

Since $$(x + h)(x + k) = x^2 + (h + k)x + hk,$$

it follows that $x^2 + bx + c$ can be factored into the binomial factors $(x + h)(x + k)$ if two numbers h and k can be found which have the sum b and the product c. The method of determining these factors is illustrated in the

EXAMPLE

Factor $x^2 - 2x - 15.$

Solution. Here $-15 = 1 \cdot -15$ or $-1 \cdot 15$ or $+5 \cdot -3$ or $+3 \cdot -5$. Of these pairs of factors of -15 only the pair -5 and $+3$ give the sum -2.

Hence $$x^2 - 2x - 15 = (x - 5)(x + 3).$$

For factoring expressions of the type $x^2 + bx + c$ we have the

Rule. *Find two numbers whose algebraic product is* $+c$ *and whose algebraic sum is* $+b$.

Write for the factors two binomials both of which have x *for their first terms and these numbers for the second terms.*

EXERCISES

Separate into binomial factors:

1. $x^2 + 5x + 6.$

2. $x^2 + 7x + 12.$

3. $x^2 + 7x + 6.$

4. $a^2 + 8a + 12.$

5. $d^2 + 11d + 18.$

6. $a^2 + 10a + 25.$

7. $d^2 - 3d - 10.$

8. $c^2 - 2c - 35.$

9. $c^2 - 4c - 12.$

10. $r^2 - r - 90$.
11. $m^2 - 3m - 18$.
12. $x^2 - 2x - 24$.
13. $9 - 10x + x^2$.
14. $r^2 - 4rs + 3s^2$.
15. $m^2 - 7mn + 10n^2$.
16. $1 - 5x + 6x^2$.
17. $1 + 2n - 24n^2$.
18. $a^2 + 9a - 10$.
19. $(a+b)^2 - 2(a+b) - 8$.
20. $(x-y)^2 + 4(x-y) + 3$.
21. $a^{2n} + 12a^n + 35$.
22. $c^{2a} - 7c^a - 18$.
23. $m^{2n} - 11m^n - 12$.
24. $a^{4x} - 5a^{2x} - 6$.
25. $b^{4y} - 2b^{2y} - 35$.

27. The general quadratic trinomial. The type form is

$$ax^2 + bx + c.$$

For many trinomials of this type two binomial factors of the form $(hx+k)(mx+n)$ may be found. The method of factoring such trinomials is illustrated in the

EXAMPLE

Factor $2x^2 + 7x - 15$.

Solution. $2x^2 + 7x - 15 = (?\,x + ?)(?\,x + ?)$.

To find the proper factors we must supply such numbers for the interrogation points in (1) and (2) as will give

$$\begin{array}{rrr} ?\,x + ? & & (1) \\ ?\,x + ? & & (2) \\ \hline 2x^2 + ?\,x & & \\ + ?\,x & - 15 & \\ \hline 2x^2 + 7x & - 15 & (3) \end{array}$$

$2x^2$ for the product of the first two terms of the binomials,
-15 for the product of the last two terms of the binomials,
$+7x$ for the sum of the cross products.

Now $\quad 2x^2 = 2x \cdot x, \qquad (4)$

and $\quad -15 = -1 \cdot +15;\ 1 \cdot -15;\ +3 \cdot -5;\ -3 \cdot +5. \qquad (5)$

The factors of 2 and -15 from (4) and (5) may be substituted for the interrogation points in (1) and (2) to form the following pairs of binomials, each having a product containing the *first* and *last* terms of the trinomial:

$$\begin{array}{cccccccc} 2x-1 & 2x+15 & 2x+1 & 2x-15 & 2x+3 & 2x-5 & 2x-3 & 2x+5 \\ x+15 & x-1 & x-15 & x+1 & x-5 & x+3 & x+5 & x-3 \end{array}$$

By trial we find that only the seventh pair has $+7x$ for the sum of its cross products, which gives the middle term of the trinomial.

Therefore $2x^2 + 7x - 15 = (2x - 3)(x + 5)$.

After a little practice it will usually be found unnecessary to write down all of the pairs of binomials that do not produce the required product.

If none of the pairs gives the required product, the given trinomial is *prime*.

If an expression of the form $ax^2 + bx + c$ is not prime, it can be factored by applying the

Rule. *Find two binomials, such that*

I. The product of the first terms is ax^2;

II. The product of the last terms is $+c$;

III. The sum of the cross products is $+bx$.

EXERCISES

Factor:

1. $2x^2 + 5x + 2$.

2. $2x^2 + 7x + 6$.

3. $2a^2 + 9a + 10$.

4. $4a^2 + 7a + 3$.

5. $3x^2 + 13x + 12$.

6. $3x^2 + 17x + 10$.

7. $4x^2 + 8x + 3$.

8. $6r^2 + 7r + 2$.

9. $2b^2 - 5b + 2$.

10. $3x^2 - 8x + 5$.

11. $6c^2 + 7c + 2$.

12. $3x^2 - 11x + 6$.

13. $3x^2 - 11x + 8$.

14. $4x^2 - 13x + 10$.

15. $10x^2 - 29x + 10$.

16. $12x^2 - 11xy + 2y^2$.

17. $2a^2 + 3a - 2$.

18. $3r^2 + r - 2$.

19. $2a^2 - a - 15$.

20. $8s^2 - 6s - 9$.

21. $9a^2 + 3a - 2$.

22. $12r^2 + 10r - 12$.

23. $10r^2 - 19rs - 15s^2$.

24. $6a^4 - 11a^3b - 7a^2b^2$.

25. $6x^3 + 10x^2y - 4xy^2$.

26. $2x^2 - 5xy - 3y^2$.

27. $2x^2 + 5xy - 12y^2$.

28. $3a^3 - 2a^2b - 8ab^2$.

29. $6a^{2n} - 7a^n + 2$.

30. $3a^{2n} - 10a^n - 8$.

31. $10a^{2n} - a^n - 3$.

28. Expressions reducible to the difference of two squares. The type form is

$$a^4 + ka^2b^2 + b^4.$$

If k has such a value that the trinomial is not a perfect square, a trinomial of this type can often be written as the *difference of two squares.* Thus, if $k = 1$, the addition and subtraction of a^2b^2 accomplishes this result.

EXAMPLES

1. Factor $a^4 + a^2b^2 + b^4$.

Solution.
$$\begin{aligned} a^4 + a^2b^2 + b^4 &= a^4 + 2\,a^2b^2 + b^4 - a^2b^2 \\ &= (a^2 + b^2)^2 - (ab)^2 \\ &= (a^2 + b^2 + ab)\,(a^2 + b^2 - ab). \end{aligned}$$

2. Factor $49\,h^4 + 34\,h^2k^2 + 25\,k^4$.

Solution. If $36\,h^2k^2$ is added, the expression becomes a perfect trinomial square. Adding and subtracting $36\,h^2k^2$, we have

$$\begin{aligned} 49\,h^4 + 34\,h^2k^2 + 25\,k^4 &= 49\,h^4 + 70\,h^2k^2 + 25\,k^4 - 36\,h^2k^2 \\ &= (7\,h^2 + 5\,k^2)^2 - (6\,hk)^2 \\ &= (7\,h^2 + 5\,k^2 + 6\,hk)\,(7\,h^2 + 5\,k^2 - 6\,hk). \end{aligned}$$

EXERCISES

Factor:

1. $x^4 + x^2y^2 + y^4$.
2. $c^4 + c^2d^2 + d^4$.
3. $a^4 + a^2b^4 + b^8$.
4. $a^4 + 3\,a^2b^2 + 4\,b^4$.
5. $m^4 + m^2 + 1$.
6. $x^4 + 5\,x^2 + 9$.
7. $c^4 + 4\,c^2 + 16$.
8. $25\,a^4 - 19\,a^2 + 1$.
9. $25\,x^4 - 11\,x^2 + 1$.
10. $4\,r^4 - 44\,r^2s^2 + 49\,s^4$.
11. $25\,x^4 - 19\,x^2 + 9$.
12. $9\,ax^8 - 28\,ax^4y^4 + 4\,ay^8$.
13. $4\,a^8x + 3\,a^4b^4x + 9\,b^8x$.
14. $4\,a^4 + 1$.

    Hint. $4\,a^4 + 1 =$ $4\,a^4 + 4\,a^2 + 1 - 4\,a^2$.
15. $c^4 + 4\,d^4$
16. $64\,a^4x^2 + x^2$.
17. $4\,a^{4n} + b^{4n}$.
18. $x^{4m} + 4\,y^{4n}$.

29. A binomial the sum or the difference of two cubes. The type form is

$$a^3 \pm b^3.$$

$a^3 + b^3$ divided by $a + b$ gives the quotient $a^2 - ab + b^2$, and $a^3 - b^3$ divided by $a - b$ gives the quotient $a^2 + ab + b^2$.

Therefore $$a^3 + b^3 = (a + b)(a^2 - ab + b^2), \quad (1)$$
and $$a^3 - b^3 = (a - b)(a^2 + ab + b^2). \quad (2)$$

Formulas (1) and (2) above may be applied as in the

EXAMPLES

1. Factor $a^3 + 27$.

Solution. $a^3 + 27 = a^3 + 3^3 = (a + 3)(a^2 - a \cdot 3 + 3^2)$
$= (a + 3)(a^2 - 3a + 9)$.

2. Factor $8 - x^3$.

Solution. $8 - x^3 = 2^3 - x^3 = (2 - x)(2^2 + 2x + x^2)$
$= (2 - x)(4 + 2x + x^2)$.

EXERCISES

Factor:

1. $x^3 + y^3$.
2. $a^3 + d^3$.
3. $a^3 + 2^3$.
4. $c^3 + 5^3$.
5. $d^3 + 8$.
6. $d^3 + 27$.
7. $x^3 + 125$.
8. $x^3 - y^3$.
9. $x^3 - 2^3$.
10. $a^3 - 27$.
11. $m^3 - 64$.
12. $m^3 - (2n)^3$.
13. $8x^3 - y^3$.
14. $r^3 - 27s^3$.
15. $64 - x^3$.
16. $125 - x^3$.
17. $a^3 + (b^2)^3$.
18. $c^3 + d^6$.
19. $m^3 - n^9$.
20. $(2a)^3 - (3b)^3$.
21. $8x^3 - 27y^3$.
22. $125x^3 + 8y^3$.
23. $(a + b)^3 + c^3$.
24. $a^3 + b^3 + a + b$.
25. $x^3 - y^3 + x - y$.
26. $m^3 - n^3 - m + n$.
27. $x^3 - 8y^3 + x - 2y$.
28. $r^3 - 8s^3 + r - 2s$.
29. $a^{3n} + b^{3n}$.
30. $a^{3m} - b^{3n}$.

30. The Remainder Theorem. If any rational integral expression in x be divided by $x-n$, the remainder is the same as the original expression with n substituted for x. This fact is illustrated in the

EXAMPLE

Divide x^2-5x+6 by $x-n$.

Solution.

$$
\begin{array}{l|l}
x^2 - 5x + 6 & x-n \\
\underline{x^2 - nx} & x+(n-5) \\
(n-5)x+6 & \\
\underline{(n-5)x \quad -n^2+5n} & \\
\qquad\qquad n^2-5n+6 = \text{Remainder} &
\end{array}
$$

Here the remainder n^2-5n+6 is the same as x^2-5x+6, the given expression, when n is substituted for x.

EXERCISES

1. Divide x^2+bx+c by $x-n$ and show that the remainder is n^2+bn+c.

2. Divide x^2+bx+c by $x-a$ and find the remainder.

3. Divide x^3+ax^2+bx+c by $x-n$ and find the remainder.

4. In $(x^3+x^2-5x+3)\div(x-2)$ find the remainder (a) by division, (b) by the Remainder Theorem.

Solution (b). $2^3+2^2-5\cdot 2+3=5$.

5. In $(x^3-x+5)\div(x-3)$ find the remainder (a) by division, (b) by the Remainder Theorem.

By use of the Remainder Theorem find the remainders in the following:

6. $(x^3+x^2-5x+8)\div(x-3)$.
7. $(x^3-3x-15)\div(x+4)$.
8. $(x^3-2x^2-100)\div(x-5)$.
9. $(x^3-2x^2-2x-3)\div(x-3)$.
10. $(x^4-2x^3+x-2)\div(x-2)$.

31. Factor Theorem. By substituting 2 for x in x^2-5x+6 we obtain $4-10+6$, or 0. Hence $x-2$ is an exact divisor (or factor) of x^2-5x+6. Again, if 3 is substituted for x in x^2-5x+6, the expression equals zero. Hence $x-3$ is a factor of x^2-5x+6. These examples illustrate the

Theorem. *If any rational integral expression in* x *becomes zero when a number* n *is substituted for* x, *then* $x-n$ *is a factor of the expression.*

The Factor Theorem may be used to factor some of the preceding exercises and, in addition, many others which are very difficult to factor by previous methods.

NOTE. By means of the Factor Theorem we are able to solve cubic and higher equations when the roots are integers. The solution of the general cubic equation is one of the famous problems of mathematics and one which is accompanied by many interesting applications. This problem was first solved by the Italian, Tartaglia, about 1530, but was published by Cardan, to whom Tartaglia explained his solution on the pledge that he would not divulge it. For many years the credit for the discovery was given to Cardan, and to this day it is usually called Cardan's Solution.

When searching for the values of x which will make an expression zero, only integral divisors of the last term of the expression (arranged according to the descending powers of x) need be tried, for the last term of the factor must be an integral divisor of the last term of the expression.

EXAMPLE

Factor x^3+2x-3.

Solution. If $x-n$ is a factor of x^3+2x-3, then n must be an integral divisor of 3. Now the factors of -3 are $1, -1, 3$, and -3. If 1 is put for x, then x^3+2x-3 equals zero, hence $x-1$ is a factor of x^3+2x-3. Dividing x^3+2x-3 by $x-1$, we obtain the quotient x^2+x+3. Since x^2+x+3 is prime, the factors of x^3+2x-3 are $x-1$ and x^2+x+3.

ORAL EXERCISES

1. Is $x-1$ a factor of x^3+3x-4?
2. Is $x-2$ a factor of $2x^3+x^2-20$?
3. Is $a-2$ a factor of a^3-3a+2?
4. Is $x-1$ a factor of x^3+3x^2-4?
5. Is $r+1$ a factor of r^3-4r^2-4r+1?
6. Is $r-3$ a factor of $2r^3-r^2+5$?
7. Is $s+1$ a factor of $3s^3-5s^2+8$?
8. Is $k-3$ a factor of $2k^3-5k^2-9$?

EXERCISES

Factor:

1. x^3+x-2.
2. x^3+2x+3.
3. a^3+a^2-36.
4. x^3+x-10.
5. d^3+d^2-12.
6. x^3-2x^2-5x+6.
7. x^3-x^2+4x-4.
8. y^3-y^2-9y+9.
9. x^4-7x^2-6x.
10. $x^3-7x^2+4x+12$.
11. $2x^3-2x^2-x-6$.
12. x^3-x^2-4.
13. $3x^3-2x^2+2x-3$.
14. $a^4-6a^3+11a^2-6a$.

32. The sum or difference of two like powers. The type form is

$$a^n \pm b^n.$$

The cases in which $a^n \pm b^n$ is divisible by $a+b$ or $a-b$ can be determined by the Factor Theorem.

Thus in a^n-b^n, n being either an odd or an even integer, substitute b for a. Then a^n-b^n becomes $b^n-b^n=0$. Therefore $a-b$ is always a factor of a^n-b^n.

In a^n-b^n, n being even, put $-b$ for a. Then a^n-b^n becomes $b^n-b^n=0$, since $(-b)^n$ is *positive* when n is *even*. Therefore when n is even $a+b$ as well as $a-b$ is an exact divisor of a^n-b^n.

In $a^n + b^n$, n being even, put either $+b$ or $-b$ for a. Then $a^n + b^n$ becomes $b^n + b^n$, which is not zero. Therefore $a^n + b^n$ is never divisible by $a + b$ or $a - b$ when n is even.

In $a^n + b^n$, n being odd, put $-b$ for a. Then $a^n + b^n$ becomes $(-b)^n + b^n = 0$, since $(-b)^n$ is *negative* when n is *odd*. Therefore when n is odd $a + b$ is a divisor of $a^n + b^n$.

Summing up:

I. $\boldsymbol{a^n - b^n}$ is *always* divisible by $\boldsymbol{a - b}$.

II. $\boldsymbol{a^n - b^n}$, when n is *even*, is divisible both by $\boldsymbol{a + b}$ and by $\boldsymbol{a - b}$.

III. $\boldsymbol{a^n + b^n}$ is *never* divisible by $\boldsymbol{a - b}$.

IV. $\boldsymbol{a^n + b^n}$, when n is *odd*, is divisible by $\boldsymbol{a + b}$.

ORAL EXERCISES

For each of the following, state a binomial factor:

1. $x^8 - y^8$.	**6.** $x^7 - y^7$.	**11.** $m^8 + 2^8$.	**16.** $1 - r^7$.
2. $x^8 - 5^8$.	**7.** $x^7 - 2^7$.	**12.** $a^8 + 8$.	**17.** $1 - r^n$.
3. $27 - a^8$.	**8.** $2^7 x^7 - y^7$.	**13.** $x^5 + y^5$.	**18.** $1 + r^7$.
4. $x^5 - 2^5$.	**9.** $a^4 - c^4$.	**14.** $r^5 + 2^5$.	**19.** $1 + r^n$.
5. $32x^5 - y^5$.	**10.** $a^8 - b^8$.	**15.** $a^5 + 32$.	**20.** $5^8 + 1$.

21. $8^8 - 1$.

22. $10^8 - 1$.

23. Is $10^7 + 1$ divisible by 11?

24. Is $10^9 - 1$ divisible by 9?

EXAMPLE

Factor $x^5 + y^5$.

Solution. By division,

$$\frac{x^5 + y^5}{x + y} = x^4 - x^3y + x^2y^2 - xy^3 + y^4.$$

Hence $$x^5 + y^5 = (x + y)(x^4 - x^3y + x^2y^2 - xy^3 + y^4).$$

Note that the signs of the second factor are alternately plus and minus. Also note the order in which the exponents occur.

EXERCISES

Factor:

1. $x^5 + z^5$.
2. $x^5 + 1$.
3. $a^5 + 2^5$.
4. $x^5 + 32$.
5. $(a^2)^5 + (b^2)^5$.
6. $x^5 - z^5$.

HINT. Find the second factor by division and observe the signs of the terms and the order in which the exponents occur.

7. $a^5 - 2^5$.
8. $a^5 - 32x^5$.
9. $(2x)^5 - 243y^5$.
10. $a^7 - x^7$.
11. $1 - r^7$.
12. $x^7 - 128$.
13. $x^{10} + y^{15}$.
14. $a^{10} + 32x^{15}$.
15. $x^7 + a^7$.
16. $1 + r^7$.
17. $128x^7 + 1$.
18. $1 - r^n$.

HINT. Write only the first five terms and the last term of the polynomial factor.

19. $1 + r^n$.
20. $x^6 - y^6$.

HINT. Factor first as the difference of two squares.

21. $x^8 - y^8$.
22. $a^{10} - b^{10}$.
23. $a^{12} - b^{12}$.

33. General directions for factoring. The following suggestions will prove helpful in factoring:

I. First look for a common monomial factor, and if there is one (other than 1), separate the expression into its greatest monomial factor and the corresponding polynomial factor.

II. Then from the form of the polynomial factor determine with which of the following types it should be classed, and use the methods of factoring applicable to that type.

1. $ax + ay + bx + by$.
2. $a^2 \pm 2ab + b^2$.
3. $a^2 - b^2$.
4. $x^2 + bx + c$.
5. $ax^2 + bx + c$.
6. $a^4 + ka^2b^2 + b^4$.
7. $a^3 \pm b^3$.
8. $a^n \pm b^n$.

III. Proceed again as in II with each polynomial factor obtained, until the original expression has been separated into its prime factors.

IV. If the preceding steps fail, try the Factor Theorem.

REVIEW EXERCISES

Factor:

1. $6x^4 + 2x^3 + 2x^2$.
2. $5a^3 + 2a^2 - 15a - 6$.
3. $a^2 + 4ab + 4b^2$.
4. $2c^3d - 8cd^3$.
5. $3m^3 - 3m^2 - 18m$.
6. $2x^2 + 3ax + a^2$.
7. $x^4 - 7x^2y^2 + 9y^4$.
8. $a^4c - ac^4$.
9. $2x^7y - 2xy^7$.
10. $x^5 - 2x^4 - 9x^2$.
11. $ac + 2bc - ad - 2bd$.
12. $18r^3 - 24r^2s + 8rs^2$.
13. $45x^3y - 20xy^3$.
14. $2h^3k + 4h^2k^2 - 30hk^3$.
15. $3a^2 - 10ab + 3b^2$.
16. $x^4 + 7x^2 + 16$.
17. $a^4d - 8ad^4$.
18. $2a^7 + 64a^2$.
19. $m^3n + mn^3$.
20. $x^3 + 4x - 5$.
21. $ax^2 - 4a + 3x^2 - 12$.
22. $a^3b^3 - 4a^2b^2 + 4ab$.
23. $a^2c^2 - c^2 - a^2 + 1$.
24. $x^3y^3 - 13x^2y^2 - 14xy$.
25. $2r^3 - 5r^2s - 3rs^2$.
26. $x^4 - 13x^2 + 36$.
27. $a^5 - a^2b^3 - a^3b^2 + b^5$.
28. $x^{12} - x^2$.
29. $x^3 - 3x^2 - 4x + 12$.
30. $x - x^2 - x^3 + x^4$.
31. $(a + x)^2 + 10(a + x) + 25$.
32. $(x + r)^2 - 3(x + r) - 18$.
33. $x^3 - 8x^2 - x + 8$.
34. $a^3 + 27a^2$.
35. $64d^3 + 2d^3$.
36. $ab^4 - ab^2c^2 + ab^3c - abc^3$.
37. $(x + y)^2 - 6(x + y)z + 9z^2$.
38. $(a - x)^2 - 16(m - n)^2$.
39. $2x^3 - 2x^2 - 12x$.
40. $10a^3c - 15a^2c^2 - 70ac^3$.
41. $a^4 - 11a^2 + 1$.
42. $a^4 + ab^{15}$.
43. $a^5 + x^{15}$.
44. $x^2(x - 1)^2 - x(x - 1)$.
45. $4(a - b)^2 - 12(a - b)c + 9c^2$
46. $a^2 - 10ab + 25b^2 - c^2$.
47. $x^2 - 2x(a - b) - 35(a - b)^2$.
48. $6x^2 - 13x + 6$.
49. $x^3 - 8x^2 + 17x - 10$.
50. $x^4 - x^2y^2 + x^3 - x^2y$.
51. $a^2 - 12a + 36 - a^4$.
52. $a^2 - b^2 + (a - b)^2$.
53. $c^2 - 2cd + d^2 - 2(c - d) - 35$.
54. $2(a + b)^2 + 3c(a + b) - 2c^2$.
55. $m^4 - 7m^2n^2 + n^4$.
56. $a^3 + b^3 + 3a^2b + 3ab^2$.

57. $32x^3 - x^3y^{10}$.

58. $5(a-b)^2 - a + b$.

59. $c^2 + 4d^2 - x^2 - 4cd$.

60. $(a-2x)c^2 + (2x-a)d^2$.

61. $x^2 - 20 + x^4$.

62. $6a^4 + 3a^3 - 3a^2$.

63. $16a^4 + 7a^2 + 1$.

64. $x^6y^6 - 64$.

65. $x^7 - 18x^5 - 27x^4$.

66. $a^7 - a^2b^5 + a^2b - a^3$.

67. $a^4 + 8ab^3 + a^2 + 2ab$.

68. $a^2 + 2a + 1 - b^2 + 2bc - c^2$.

69. $9x^2 - 4y^2 - 3x - 2y$.

70. $x^2 - 5(2x - 5)$.

71. $a^3 - 8 - 7a^2 + 14a$.

72. $3a^2 - 14a(b-c) + 8(b-c)^2$.

73. $a^4x^4 + 4$.

74. $x^3y^6 + x^3z^6$.

75. $x^3 - 10x - 3$.

76. $x^8 - x^4y^4 - x^4(x^2 - y^2)$.

77. $a^8 - 5a^4 + 4$.

78. $a^4 - 9a^2 - a + 3$.

79. $x - 1 + x^5 - x^3$.

80. $x^3 + x - y - y^3$.

81. $x^2 + 2xy + y^2 - 25a^2 - 10a - 1$.

82. $a^4 - b^4 - 2ab(a^2 - b^2)$.

83. $2x^3 + 3x^2 + x$.

84. $m^7 - 8m - 7m^4$.

85. $3a^3 - 9a(2a - 3)$.

86. $2x^4 - 10x^2 + 4x$.

87. $10a - 7a^2 - 6a^3$.

88. $a^3 + 1 + 3a^2 + 3a$.

89. $12x^2 - 8x + x^4 - 6x^3$.

90. $a^nx^2 + 2a^nx + a^n$.

91. $mr - ms - nr + ns$.

92. $h^{2m} - 2h^mk^n + k^{2n}$.

93. $a^{2m} - b^{2n}$.

94. $x^{2n} + (r+s)x^n + rs$.

95. $hrx^{2n} + hrx^n + hsx^n + hs$.

96. $a^{4m} + a^{2m}b^{2n} + b^{4n}$.

97. $a^{3m} + b^{3n}$.

98. $a^{3m} - b^{3n}$.

99. $a^{5m} + b^{5n}$.

100. $a^{5m} - b^{5n}$.

101. Solve for x, $ax + bx - 3a = 3b$.

Solution. Rewriting, $\quad ax + bx = 3a + 3b.$

Factoring in each member, $\quad x(a+b) = 3(a+b).$

Dividing each member by $a + b$, $\quad x = 3.$

102. Solve for x, $cx + dx = c^2 + cd$.

103. Solve for m, $ma - mc + bc = ab$.

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104. Solve for y, $5ay - 3by - 5a^2 + 3ab = 0$.

105. Solve for z, $az - 3ad = bz - 3bd$.

106. Solve for x, $cx - 2dx = c^2 - 4d^2$.

107. Solve for m, $am - m + 1 - a^2 = 0$.

108. Solve for y, $by + 3dy - b^2 + 9d^2 = 0$.

109. Solve for r, $r(2a - 7c) = 4a^2 - 28ac + 49c^2$.

110. Solve for m, $m(3a - c) - 9ad = 2ec - 6ae - 3cd$.

111. Solve for x, $ax - 3bx - a^2 = 3b^2 - 4ab$.

112. Solve for s, $2as - 7s + 13a = 2a^2 + 21$.

113. Solve for t, $2te - 2e + 15 - 8e^2 + 3t = 0$.

114. Solve for y, $y + 21d^2 + 4d = 7dy + 1$.

115. Solve for x, $d(1 - 3a) + x + 3ac = c + 3ax$.

116. Solve for z, $az + ae - 2ec + 2cd = 2cz + ad$.

117. Solve for y, $a - 3 = ay - 3y - 2(a - 3)$.

118. Solve for x, $5x - 2cx - a(5 - 2c) = 5a + 5 - 2c - 2ac$.

119. Solve for x, $a^2x - ax + x - a^3 - 1 = 0$.

120. Solve for x, $c^3 - c^2x - 2cdx - 8d^3 = 4d^2x$.

121. Solve for m, $m(a - 2)(a^2 + 4) = a^4 - 16$.

122. Solve for s, $16s - 8as + 4a^2s - 2a^3s + a^4s - 32 = a^5$.

34. Solution of equations by factoring. The methods of factoring enable us to solve many equations in one unknown. In the solution of equations by factoring use is made of the

Principle. *If the product of two or more factors is zero, one of the factors must be zero.*

Each of the factors may be zero, but the vanishing of one factor is sufficient to make the product zero.

XAMPLE

$' = 6x.$

have

$^2 - 6x = 0.$

$- 3) = 0.$

m setting each factor separately

$+ 3.$

iginal equation, we find that

n.

example is stated in the

ation so that the second
nember, set each factor
, and solve the result-

nber of $x^3 - x^2 = 6x$
y the roots $- 2$ and
e root 0.
d by the rule, the
$5x + 6 = x - 2$
has the root 4
e been lost if
and the equa-
an equation

9. $m^2 - am - 2a^2 = 0$.
10. $x^2 - 2ax - 8a^2 = 0$.
11. $y^3 - y + 2 = 2y^2$.
12. $x^2 - 9a^2 - x + 3a = 0$.
13. $x^3 - ax^2 - 12a^2x = 0$.
14. $x^3 - a^2x + 2ax^2 - 2a^3 = 0$.
15. $x^3 + x^2 - a^2x - ax = 0$.
16. $4y + 19y^2 - 5y^3 = 0$.
17. $y^3 - 7y - 6 = 0$.
18. $y^4 - 13y^2 + 36 = 0$.
19. $x^5 - 5x^3 = -4x$.
20. $x^4 - 7x^2 = -6x$.
21. $3r^3 + 24r^2 + 48r = 0$.
22. $z^3 + 12z - 6z^2 - 8 = 0$.
23. $3x^3 + 72x + 33x^2 = 0$.
24. $5s^2 + 12s - 3s^3 = 0$.
25. $8x^2 + 71x^3 - 9x^4 = 0$.
26. $(2x - 3)^2 - (5x + 6)^2 = 0$.
27. $(x - r)^2 - (x - s)^2 = 0$.
28. $(x - 3)^2 - 2(x - 3) = 8$.
29. $x^3 + 50 - 25x - 2x^2 = 0$.
30. $2y^3 - 2y^2 - 8y + 8 = 0$.
31. $x^3 - ax^2 - 4a^2x + 4a^3 = 0$.
32. $x^3 + 8 + 6x^2 + 12x = 0$.
33. $y^3 + 3y^2 + 3y + 1 = 0$.
34. $acx^2 + bcx + adx + bd = 0$.
35. $x^2 - 7x - 8 = x + 1$.
36. $4x^2 + x - 1 = 2x - 1$.
37. $3x^2 - 11x + 6 = 3x - 2$.
38. $a^2x^2 - 2ax - 3 = ax - 3$.
39. $x^3 - 2x^2 - 15x = x^2 - 5x$.
40. $6x^2 + 7x - 3 = 3x - 1$.

35. The highest common factor. The **highest common factor** (H.C.F.) of two or more monomials or polynomials is the expression of highest degree, with the greatest numerical coefficient, which is an exact divisor of each.

Thus the H.C.F. of $28a^2b^3$ and $42a^2b^2$ is $14a^2b^2$. The H.C.F. of $x^3 - 4x$ and $x^3 - 5x^2 + 6x$ is $x(x - 2)$, or $x^2 - 2x$.

EXAMPLE

Find the H.C.F. of $9x^4 - 36x^2$ and $3x^7 - 12x^6 + 12x^5$.

Solution. Factoring, we have

$$9x^4 - 36x^2 = 3^2x^2(x + 2)(x - 2),$$

$$3x^7 - 12x^6 + 12x^5 = 3x^5(x - 2)^2.$$

Therefore the H.C.F. is $3x^2(x - 2)$, which equals $3x^3 - 6x^2$.

The method used in the preceding solutions for finding the H.C.F. of two or more monomials or polynomials is stated in the

Rule. *Separate each expression into its prime factors. Then find the product of such factors as occur in each expression, using each prime factor the least number of times it occurs in any one expression.*

If two or more polynomials have no common factor other than 1, then 1 is their H.C.F., and the polynomials are said to be prime to each other.

EXERCISES

Find the H.C.F. of the following:

1. 12, 18, 24.
2. 15, 25, 40.
3. 24, 60, 72.
4. $12a^2$, $30a^5$, $36a^4$.
5. $30c^2d$, $45cd^2$, $75c^2d^2$.
6. $28a^2b^4$, $42ab^5$, $70a^2b^3$.
7. $66c^4x$, $132c^2x^3$, $165c^4x^2$.
8. $a^2+2ab+b^2$, a^2-b^2.
9. $3a^2-3b^2$, $9(a-b)^2$, $3a^3-3b^3$.
10. $ax^2-2axy+ay^2$, $a^2x^2-a^2y^2$, $2ax^3-2ay^3$.
11. $2a^2m^2-2a^2n^2$, $4am^2-12amn+8an^2$, $10am-10an$.
12. $25x^3-25x^2y$, $10x^3-20x^2y-30xy^2$, $5x^4-5xy^3$.
13. $3x^4-6x^3y$, $6x^6-24x^4y^2$, $12x^5-96x^2y^3$.
14. $24a^6-6a^4b^2$, $48a^6+24a^5b$, $48a^5-48a^4b-36a^3b^2$.
15. $5x^7-160x^2$, $15x^5-60x^3$, $25x^7-200x^4$.
16. $18a^4-2a^2b^2$, $12a^6-8a^5b-4a^4b^2$, $30a^4b^2+10a^3b^3$.
17. $4x^4-4x^2y^2$, $5x^7-5x^3y^4$, $8x^{11}-8x^5y^6$.
18. $a^5-3a^4b+2a^3b^2$, $a^5-2a^4b-a^3b^2+2a^2b^3$, $a^7-5a^5b^2+4a^3b^4$.

NOTE. The most famous, and in some respects the most perfect, treatise on elementary mathematics ever written is Euclid's "Elements." About one third of the material of the thirteen books treats

topics which to-day would be considered arithmetical in character. In appearance and language, however, they are all geometrical, for Euclid represents quantities not by numerals, as we do in arithmetic, or by letters, as we do in algebra, but by lines. Book VII contains the earliest statement of a general method for finding the G. C. D. of two numbers. This method, though never necessary in elementary mathematical work, is so perfect and beautiful from a scientific point of view that until recently it remained in elementary treatises on algebra and arithmetic by force of tradition. It is a great tribute to Euclid's genius that he was able to devise so perfect a method for the process that all the efforts of two thousand years have been unable to improve it essentially. It is of fundamental importance in advanced portions of algebra.

CHAPTER IV

FRACTIONS

36. Operations on fractions. The change of a fraction to higher or to lower terms, and the addition and the subtraction of fractions in both arithmetic and algebra, depend on the

Principle. *The numerator and the denominator of a fraction may be multiplied by the same expression or divided by the same expression without changing the value of the fraction.*

Thus $\frac{3}{4} = \frac{3 \cdot 4}{4 \cdot 4} = \frac{12}{16}$, and $\frac{18}{30} = \frac{18 \div 6}{30 \div 6} = \frac{3}{5}$.

Similarly, $\frac{a}{b} = \frac{a \cdot n}{b \cdot n} = \frac{an}{bn}$, and $\frac{a}{b} = \frac{a \div n}{b \div n} = \frac{\frac{a}{n}}{\frac{b}{n}}$.

It should be noted that by the application of this principle a fraction is changed in form but not in value.

ORAL EXERCISES

Divide both numerator and denominator of

1. $\frac{5}{15}$ by 5.
2. $\frac{6x}{15}$ by 3.
3. $\frac{2x}{3x^2}$ by x.
4. $\frac{12x}{8x^3}$ by $4x$.
5. $\frac{9x^2}{15x^5}$ by $3x^2$.
6. $\frac{15a^3}{12a}$ by $3a$.
7. $\frac{5(a-2)}{3(d-2)}$ by $a-2$.
8. $\frac{x+2}{x^2-4}$ by $x+2$.
9. $\frac{3x-6}{x^2-4}$ by $x-2$.

Reduce to lower terms:

10. $\dfrac{2\,ax}{3\,a}$.

11. $\dfrac{5\,x^2y}{10\,xy}$.

12. $\dfrac{2(x-5)}{3\,x(x-5)^2}$.

13. $\dfrac{3(a-2)}{a^2-4}$.

14. $\dfrac{5\,x+10}{2\,x+4}$.

15. $\dfrac{x^3+27}{x(x+3)}$.

16. $\dfrac{x^2+4}{x^4-16}$.

17. $\dfrac{x^3+8}{x^2+4\,x+4}$.

18. $\dfrac{4-2\,x+x^2}{x^3+8}$.

EXERCISES

Reduce to lowest terms:

1. $\dfrac{2\,a^3b^2}{10\,a^2b^3}$.

2. $\dfrac{15\,xy^3}{245\,x^3y}$.

3. $\dfrac{x^2-9}{2\,x^2+6\,x}$.

4. $\dfrac{5\,x^2+5\,xy}{25\,x^4-25\,x^2y^2}$.

5. $\dfrac{2\,a^3+8\,a^2b+8\,ab^2}{5\,a^5-20\,a^3b^2}$.

6. $\dfrac{48\,x^4-6\,xy^3}{32\,x^4-32\,x^3y+8\,x^2y^2}$.

7. $\dfrac{3\,x^3-375}{2\,x^2+10\,x+50}$.

8. $\dfrac{a^3+6\,a^2b+9\,ab^2}{3\,a^2b+9\,ab^2}$.

9. $\dfrac{2\,c^2+2\,cd-4\,d^2}{5\,c^2-5\,d^2}$.

10. $\dfrac{2\,c^6-64\,cd^5}{16\,c^4-40\,c^3d+16\,c^2d^2}$.

11. $\dfrac{ad-3\,ax+2\,cd-6\,cx}{ad-3\,ax-cd+3\,cx}$.

12. $\dfrac{3\,x^3-9\,x^2-30\,x}{2\,x^3-8\,x^2+8\,x}$.

13. $\dfrac{4\,x^2+6\,x-40}{4\,x-15\,a+6\,ax-10}$.

14. $\dfrac{5\,x^4-40\,x}{3\,x^5-96}$.

15. $\dfrac{12\,a^3+10\,a^2-12\,a}{4+9\,a^2-12\,a}$.

16. $\dfrac{3\,x^4y+3\,x^2y^3+3\,y^5}{5\,x^3y+5\,xy^3+5\,x^2y^2}$.

17. $\dfrac{x^2-6\,x+9-4\,a^2}{x^2+4\,ax+4\,a^2-9}$.

18. $\dfrac{32\,x^2-x^7}{32+16\,x+4\,x^3+8\,x^2+2\,x^4}$.

19. $\dfrac{(\frac{2}{3})^2-1}{\frac{2}{3}+1}$.

20. $\dfrac{(\frac{3}{5})^3-1}{\frac{3}{5}-1}$.

21. $\dfrac{(\frac{3}{2})^2-(\frac{3}{2})(\frac{2}{5})+(\frac{2}{5})^2}{(\frac{3}{2})^3+(\frac{2}{5})^3}$.

37. Changes of sign in a fraction. In the various operations on fractions three signs must be considered: the sign of the numerator, the sign of the denominator, and the sign before the fraction. Since a fraction is an indicated quotient, the law of signs in division, when considered in connection with the sign of a fraction, gives the following identity:

$$+\frac{+a}{+b}=+\frac{-a}{-b}=-\frac{-a}{+b}=-\frac{+a}{-b}.$$

From the above we have the

Principle. *In a fraction the signs of both numerator and denominator, or the sign of the numerator and the sign before the fraction, or the sign of the denominator and the sign before the fraction, may be changed without altering the value of the fraction.*

In applying this principle to a polynomial denominator or numerator, like $3x^2-5x+2$, we change its sign by changing the signs of every term of the polynomial. If the polynomial denominator is in factored form, like $(x-3)(2x-1)(3x+1)$, its sign is changed by changing the sign of *any* one factor or of an *odd number* of factors.

ORAL EXERCISES

Read the following fractions in three additional ways, using the above principle:

1. $\frac{-a}{x}$. **2.** $\frac{2}{-x}$. **3.** $\frac{a}{2-x}$. **4.** $-\frac{3a}{1-a}$.

Change the sign of the following indicated products:

5. $(x-3)(x+3)$. **6.** $(2x+5)(3x-5)$. **7.** $(x-a)(x-b)$.

Change in two ways the sign of the following:

8. $(x+1)(x-1)(x^2+1)$. **10.** $(2x-3)(3x-4)(4x-5)$.

9. $(x-a)(x-b)(c-x)$. **11.** $(x-2)^2(x-1)(x+2)$.

[redacted]

12. Find the indicated product in Exercise 8. Then change the sign of two factors and find the product. Compare the results.

13. Make a general statement of which the result of Exercise 12 is an illustration.

Change the form of the following fractions so that each will contain the factor $x-1$ in its denominator:

14. $\frac{1}{1-x}$.

15. $-\frac{2}{1-x^2}$.

16. $-\frac{-3}{1-x}$.

17. $\frac{a-2}{x(1-x)^3}$.

18. $+\frac{3a}{2-x-x^2}$.

19. $-\frac{2x-3}{-1+2x-x^2}$.

By proper changes of sign make the denominators of the following fractions as nearly alike as possible:

20. $\frac{3}{(x-2)(x+2)}, \quad \frac{x+3}{(2-x)(2+x)}$.

21. $\frac{a}{(a-b)(c-a)(b-c)}, \quad \frac{b}{(b-a)(c-a)(b-c)},$

$$\frac{c}{(a-b)(a-c)(c-b)}.$$

38. Lowest common multiple. The **lowest common multiple** (L.C.M.) of two or more rational integral expressions is the expression of lowest degree, with the least numerical coefficient, which will exactly contain each.

Thus the L.C.M. of 6, 8, and 12 is 24, and the L.C.M. of a^2c, ac^3, and $2\,abc^2$ is $2\,a^2bc^3$.

EXAMPLE

Find the L.C.M. of x^3-4x, $2x^3-4x^2$, and $4x^4-8x^3$.

Solution. Factoring,

$$x^3-4x = x(x+2)(x-2),$$
$$2x^3-4x^2 = 2x^2(x-2),$$
$$4x^4-8x^3 = 4x^3(x-2).$$

The L.C.M. is $4x^3(x+2)(x-2)$.

For such expressions as can be readily factored the method of finding the L.C.M. is stated in the

Rule. *Separate each expression into its prime factors. Then find the product of all the different prime factors, using each factor the greatest number of times it occurs in any one expression.*

ORAL EXERCISES

Find the L.C.M. of the following:

1. 4, 8, 12.
2. 25, 50, 75.
3. $4x$, $6x^2$, $8x$.
4. $2ab^2$, $4a^2b$, $6ab^3$.
5. $x-2$, x^2-4, $2(x+2)$.
6. x^3-8, $2(x-2)$, x^2+2x+4.
7. x^2-9, x^2-6x+9.
8. $x-5$, x^2-x-20.
9. a^2-4, a^2-4a+4, $3(a+2)$.
10. a^2-2a+1, $1-a$, a.
11. x^3-125, $x^2+5x+25$, $5-x$.
12. $ax(a^2-x^2)$, $a^2(x^2-a^2)$, $x^2(a+x)$.

EXERCISES

Find the L.C.M. of the following:

1. a^3-ab^2, $a^2-2ab+b^2$, a^2+ab.
2. $ax+ay+bx+by$, a^2-b^2, x^2-y^2.
3. $c^2-8cd+16d^2$, c^2-16d^2, c^2+4cd.
4. $r^2-4rs-21s^2$, r^2-49s^2, r^2-9s^2.
5. $2m^2+mn-10n^2$, $4m^2-25n^2$, m^3-8n^3.
6. $8r^2-8r^4$, $1-2r+r^2$, r^2-1.
7. $125-n^3$, n^3-25n, $50n+10n^2+2n^3$.
8. a^5-32, $4-a^2$, $5a^2+10a$, $5a-10$.
9. x^3-7x+6, x^2-3x+2, x^2+2x-3.
10. $x^3+4x^2-4x-16$, x^2-4, x^2+2x-8.
11. x^4-x, x^2-x^4, x^3+x^2+x, x^3-x.
12. $4a^2-20a+25$, $25-4a^2$, $2a^2+15a+25$.

39. Equivalent fractions. Two fractions are **equivalent** if one can be obtained from the other either by multiplying or by dividing both numerator and denominator by the same expression.

Two fractions having unlike denominators cannot be added or subtracted until they have been reduced to respectively equivalent fractions having like denominators.

EXAMPLE

Reduce to respectively equivalent fractions having the lowest common denominator (L.C.D.):

$$\frac{2}{3x},\quad \frac{5}{x^2-4},\quad \text{and}\quad \frac{2x}{x^2-2x}.$$

Solution. Rewriting with denominators in factored form, we have

$$\frac{2}{3x},\quad \frac{5}{(x+2)(x-2)},\quad \text{and}\quad \frac{2x}{x(x-2)}.$$

The L.C.D. is $3x(x+2)(x-2)$.

Then $$\frac{2}{3x}=\frac{2(x+2)(x-2)}{3x(x+2)(x-2)},$$

$$\frac{5}{(x+2)(x-2)}=\frac{5\cdot 3x}{3x(x+2)(x-2)},$$

and $$\frac{2x}{x(x-2)}=\frac{2}{x-2}=\frac{2\cdot 3x(x+2)}{3x(x+2)(x-2)}.$$

To change two or more fractions (in their lowest terms) to respectively equivalent fractions having the L.C.D. we have the

Rule. *Rewrite the fractions with their denominators in factored form.*

Find the L.C.M. of the denominators of the fractions.

Multiply the numerator and the denominator of each fraction by those factors of this L.C.M. which are not found in the denominator of the fraction.

EXERCISES

Change to respectively equivalent fractions having the L. C. D.:

1. $\dfrac{2}{3x}, \dfrac{5}{2x^2}$.

2. $\dfrac{2x}{x-2}, \dfrac{3}{x+2}$.

3. $\dfrac{3}{a^2-9}, \dfrac{2a}{a-3}$.

4. $\dfrac{3a}{(a-2)^2}, \dfrac{2a^2}{a-2}$.

5. $\dfrac{2x-1}{x^2-6x+9}, \dfrac{-x+5}{-2x+6}$.

6. $\dfrac{x+1}{x^2-5x+6}, \dfrac{-x}{-x^2+4}$.

7. $\dfrac{2a}{2a^2+3a+1}, \dfrac{5a}{2a+1}$.

8. $\dfrac{2a}{a+2}, \dfrac{a+3}{3a+1}, \dfrac{2a^2+5}{3a^2+7a+2}$.

9. $\dfrac{a}{2x}, \dfrac{2a}{3x}, \dfrac{5a}{x^2+2x}$.

10. $\dfrac{2x+1}{x^2-2x+4}, \dfrac{5}{x^3+8}, \dfrac{1}{x+2}$.

11. $\dfrac{2x-9}{x^5-32}, \dfrac{-2}{-x^3+2x^2}, \dfrac{x-2}{x^5+2x^4+4x^3+8x^2+16x}$.

12. $\dfrac{ax-bx-ar+br}{ax+bx+ar+br}, \dfrac{x^2-r^2}{a^2-b^2}, \dfrac{a^2-b^2}{x^2-r^2}$.

13. $\dfrac{2x}{x^3-8}, \dfrac{5x}{2-x}, \dfrac{4}{4-x^2}, \dfrac{3x+2}{x^3+2x^2+4x}$.

40. Addition and subtraction of fractions. To find the **algebraic** sum of two or more fractions in their lowest terms we proceed as in the

EXAMPLE

Find the algebraic sum of $\dfrac{2b}{a} + \dfrac{3b}{a^2} - \dfrac{5}{ab}$.

Solution. The L.C.D. is a^2b.

Rewriting with common denominators, and adding numerators, we have

$$\frac{2ab^2}{a^2b} + \frac{3b^2}{a^2b} - \frac{5a}{a^2b} = \frac{2ab^2+3b^2-5a}{a^2b}.$$

Check. Substituting 2 for a and 3 for b, we have

$$\frac{6}{2}+\frac{9}{4}-\frac{5}{6}=\frac{36+27-10}{12},$$

$$\frac{53}{12}=\frac{53}{12}.$$

For finding the algebraic sum of two or more fractions we have the

Rule. *Reduce the fractions to respectively equivalent fractions having the lowest common denominator. Write in succession over the lowest common denominator the numerators of the equivalent fractions, inclosing each polynomial numerator in a parenthesis preceded by the sign of the corresponding fraction.*

Rewrite the fraction just obtained, removing the parentheses in the numerator.

Then combine like terms in the numerator and, if necessary, reduce the resulting fraction to its lowest terms.

Check. *Set the original expression equal to the final result and substitute in each member numerical values for the letters involved. The equation should be an identity.*

EXERCISES

Find the algebraic sum of:

1. $\frac{2x}{5}+\frac{3x}{10}-\frac{2x}{15}$.

2. $\frac{2c}{3a}-\frac{5c^2}{6a^2}+\frac{c}{9a}$.

3. $\frac{a-b}{ab}-\frac{a-3b}{a^2}-\frac{3a}{b^2}$.

4. $\frac{2rs^3+r}{r^2s}-\frac{r-6s}{3rs^2}-\frac{s^2}{2r}$.

5. $\frac{5a}{a-3}-\frac{2}{3}$.

6. $\frac{c}{c-4}-\frac{2c}{c+3}$.

7. $\frac{a}{a-5}+\frac{2a}{a+4}-\frac{1}{2}$.

8. $\frac{3x-1}{x^2-2x+3}-\frac{3}{x-2}$.

9. $\frac{2x+1}{x^2-9}-\frac{4}{x-3}$.

10. $\frac{a-3}{a^2-4}-\frac{a-3}{2-a}+\frac{5}{a+2}$.

11. $\dfrac{c^2+2}{c^2-2c-8}-\dfrac{c-5}{4-c}$.

12. $\dfrac{2c-3}{c(c-2)}-\dfrac{c^2-7}{8-c^3}$.

13. $\dfrac{m}{m-1}-\dfrac{2}{m}-\dfrac{5m-2}{m^2-1}$.

14. $2+\dfrac{5x}{x-7}$.

HINT. $2+\dfrac{5x}{x-7}=\dfrac{2}{1}+\dfrac{5x}{x-7}$.

15. $x+3-\dfrac{x^2}{x-3}$.

16. $x^2+2x+4-\dfrac{x^3+8}{x-2}$.

17. $a^2+ab+b^2-\dfrac{a^3}{a-b}$.

18. $a-\dfrac{a-6}{2a-3}-2$.

19. $c^2-\dfrac{c^4+d^4}{c^2+cd+d^2}-cd+d^2$.

20. $\dfrac{1}{x-a}-\dfrac{2x}{a^2-x^2}-\dfrac{4ax}{x^3-a^3}$.

21. $\dfrac{x+2}{2x-1}-\dfrac{x}{1-x}-\dfrac{3x^2+2x-2}{2x^2-3x+1}$.

22. $\dfrac{c+2d}{c^2-2cd+d^2}-\dfrac{1}{d-c}-\dfrac{2}{c+2d}$.

23. $\dfrac{a-3}{a^2-2a-3}-\dfrac{2-a}{a^2-3a+2}-\dfrac{1}{1-a^2}$.

24. $\dfrac{a^2-16a}{4-a^2}-\dfrac{3+2a}{a-2}+\dfrac{3a-2}{a+2}$.

25. $\dfrac{a+2}{a^2-7a+12}-\dfrac{a+2}{6+a^2-5a}-\dfrac{1}{6a-a^2-8}$.

26. $\dfrac{b+a}{b-a}-2\left(\dfrac{b}{a}-\dfrac{b}{a-b}\right)$.

27. $b-2a\left(1-\dfrac{2c-b}{c-b}\right)$.

28. $\dfrac{a+b}{(b-c)(c-a)}+\dfrac{b+c}{ac-a^2-bc+ab}+\dfrac{a+c}{(a-b)(b-c)}$.

29. $\dfrac{x^2-(m-n)^2}{(x+n)^2-m^2}+\dfrac{m^2-(x-n)^2}{(x+m)^2-n^2}+\dfrac{n^2-(x-m)^2}{(m+n)^2-x^2}$.

41. Multiplication of fractions. In algebra, as in arithmetic, the product of two or more fractions is the product of their numerators divided by the product of their denominators.

Thus $\frac{3}{4} \cdot \frac{5}{7} = \frac{15}{28}$.

Similarly, $\frac{a}{b} \cdot \frac{c}{d} = \frac{ac}{bd}$,

and $5 \cdot 1\frac{3}{7} = \frac{5}{1} \cdot \frac{10}{7} = \frac{50}{7}$.

In like manner $n \cdot \frac{a}{b} = \frac{n}{1} \cdot \frac{a}{b} = \frac{na}{b}$.

Integral and mixed expressions are reduced to fractional form before the multiplication is performed. Factors common to any numerator and any denominator are canceled the same number of times from each.

EXAMPLE

Multiply $20\,a^2\left(1 - \frac{4}{a^2}\right)\left(\frac{6\,a}{5\,a^3 - 10\,a^2}\right)$.

Solution. Writing the above in fractional form, we have

$$\frac{20\,a^2}{1} \cdot \frac{a^2 - 4}{a^2} \cdot \frac{6\,a}{5\,a^3 - 10\,a^2}.$$

Factoring and canceling,

$$\frac{\overset{4}{\cancel{20\,a^2}}}{1} \cdot \frac{(a+2)\cancel{(a-2)}}{\underset{a}{\cancel{a^2}}} \cdot \frac{6\,\cancel{a}}{\cancel{5\,a^2(a-2)}} = \frac{24\,(a+2)}{a}.$$

To find the product of two or more fractions or mixed expressions we have the

Rule. *If there are integral or mixed expressions, reduce them to fractional form.*

Separate each numerator and each denominator into its prime factors.

Cancel the factors (factor for factor) common to any numerator and any denominator.

Write the product of the factors remaining in the numerator over the product of the factors remaining in the denominator.

Check. *Set the given expression equal to the final result and substitute in each member numerical values for the letters. Simplify each member. The result should be an identity.*

42. Division of fractions. For division of fractions we have the

Rule. *Reduce all integral or mixed expressions to fractional form.*

Then invert the divisor or divisors and proceed as in multiplication of fractions.

Check as usual.

EXERCISES

Perform the indicated operations:

1. $\frac{6a}{25x^2y} \cdot \frac{10xy^2}{4a^2x}$.

2. $5c \cdot \frac{3d}{10cd^2} \cdot \frac{4c}{9d}$.

3. $\frac{7ab^2}{5cd^2} \cdot \frac{30c^3d}{14a^2b}$.

4. $\frac{8a^2c}{7c^4} \cdot \frac{21c^2}{6bc^3} \cdot \frac{20a^2b^2}{5a^3c^3}$.

5. $4\frac{1}{2} \cdot \frac{2a^2}{3x^3} \cdot \frac{x^2}{15a}$.

6. $10a \cdot \frac{7x}{2a^4} \cdot \frac{a}{21x^3}$.

7. $\frac{4m^3n}{9n^3s^2} \cdot \frac{6ns}{4m^2s} \cdot \frac{2m^2s^3}{2mn^4}$.

8. $\frac{a^2-4}{2a^2} \cdot \frac{4a}{2a-4}$.

9. $\frac{3x^2}{x+3} \cdot \frac{x^2-9}{6x^3}$.

10. $\frac{a^2-1}{2a} \cdot \frac{6a^3}{a^2-2a+1} \cdot \frac{5a-5}{3(1-a)}$.

11. $\left(2a+\frac{1}{a}\right)\left(\frac{a^2}{4a^4-1}\right)\left(a-\frac{1}{2a}\right)$.

12. $\frac{6}{35} \div \frac{3}{14}$.

13. $\frac{35}{42} \div \frac{25}{30}$.

14. $\frac{14x}{15ax^2} \div \frac{84a}{60a^2x^3}$.

15. $\frac{120c^2d}{42c^3x} \div \frac{100cd^3}{147c^2x^2}$.

16. $\dfrac{24x^2}{99xy^2}\cdot\dfrac{8y^2z}{22x^2y}\div\dfrac{48x^3z}{121x^2y^3}.$

17. $\dfrac{15rs^2}{14s^2t}\div\dfrac{300rt^3}{98r^2s}\cdot\dfrac{160st^4}{28r^2s^3}.$

18. $\dfrac{22m^2n^2}{35x^3}\div\dfrac{52m^5n}{125x^2}\div\dfrac{275n^2x}{39m^3x}.$

19. $\dfrac{5x^2}{x-y}\cdot\dfrac{(x-y)^2}{20x^2}\div\dfrac{1}{x(x-y)}.$

20. $\dfrac{x+3}{x^2-4}\cdot\dfrac{x^2-4x+4}{2x+6}\div\dfrac{x^2-2x}{x^2+2x}.$

21. $\dfrac{x^2+2x+4}{x^2-2x+4}\div\dfrac{x^3-8}{x^2+2x}\div\dfrac{x^2-2x}{x^3+8}.$

22. $\dfrac{2x^2+9x+9}{2x^2-9x+9}\div\dfrac{x^2-9}{4x^3-9x}\cdot\dfrac{x^2-6x+9}{2x^2-3x}.$

23. $\dfrac{x^3-y^3}{x^3+y^3}\div\dfrac{x^4+x^2y^2+y^4}{x+y}\cdot\dfrac{x^2-xy+y^2}{x-y}.$

24. $\dfrac{a^5+32}{a^5-32}\div\dfrac{a^3+2a^2}{a^2-2a}\cdot\dfrac{a^4+2a^3+4a^2+8a+16}{a^4-2a^3+4a^2-8a+16}.$

25. $\dfrac{x^3-6x^2+11x-6}{x^4-1}\div\dfrac{x^2-5x+6}{x^2+2x+1}\cdot\dfrac{x^2+1}{x+1}.$

26. $\dfrac{2x+5}{x^3-6x^2+18x-27}\div\dfrac{25-4x^2}{x^3+27}\div\dfrac{2x^2+x-15}{x-3}.$

27. $\left(\dfrac{2x}{3}\right)^2\left(\dfrac{5}{4x}\right)^3\left(\dfrac{6x}{25}\right)^2.$

28. $\left(\dfrac{3a}{2b}\right)^3\div\left(\dfrac{3a^2x}{5b^2}\right)\div\dfrac{(10x)^3}{64}.$

29. $\left(\dfrac{3r^2}{2s}\right)^4-4\left(\dfrac{3r^2}{2s}\right)^3\left(\dfrac{2s^2}{3r}\right)+6\left(\dfrac{3r^2}{2s}\right)^2\left(\dfrac{2s^2}{3r}\right)^2.$

30. $\left(\dfrac{2a}{5x^2}\right)^3-3\left(\dfrac{2a}{5x^2}\right)^2\left(\dfrac{5x^3}{4a^2}\right)+3\left(\dfrac{2a}{5x^2}\right)\left(\dfrac{5x^3}{4a^2}\right)^2.$

31. $\left(\dfrac{a^2}{2b}\right)^6+6\left(\dfrac{a^2}{2b}\right)^5\left(\dfrac{2b}{a}\right)+15\left(\dfrac{a^2}{2b}\right)^4\left(\dfrac{2b}{a}\right)^2+30\left(\dfrac{a^2}{2b}\right)^3\left(\dfrac{2b}{a}\right)^3.$

32. $\left(x+2+\dfrac{1}{x}\right)\left(\dfrac{x^2}{x^2-1}\right)\left(\dfrac{x-1}{x^2+x}\right).$

33. $\dfrac{1-a^3}{1+a^2}\div\left(1-\dfrac{2a^6}{a^6+1}\right)\div\left(\dfrac{1-a^2+a^4}{a^3+1}\right).$

34. $\left(x^2-2x+4-\dfrac{16}{x+2}\right)\left(\dfrac{x^2-4}{2x^2+4x+8}\right)\div\left(\dfrac{x-2}{2}\right)^2.$

35. $\dfrac{a}{a^2-4} \div \left(\dfrac{a}{3a^2+5a+2}\right)\left(3a-\dfrac{2}{a}-5\right)$.

36. $\dfrac{x+1+\dfrac{4}{x+3}}{x-5+\dfrac{12}{x+3}}$.

HINT. An expression of this form is called a complex fraction. It is simply another way of writing

$$\left(x+1+\frac{4}{x+3}\right) \div \left(x-5+\frac{12}{x+3}\right).$$

37. $\dfrac{2+\dfrac{x+6}{x-2}}{3+\dfrac{5}{x-2}}$.

38. $\dfrac{x-1-\dfrac{3}{x}}{x+1+\dfrac{x}{x-3}}$.

39. $\dfrac{\dfrac{a}{b}+1+\dfrac{b}{a}}{\dfrac{b^2}{a^2}+1+\dfrac{a^2}{b^2}}$.

40. $\dfrac{\dfrac{a^2}{b^2}+1+\dfrac{b^2}{a^2}}{1-\dfrac{a^2+b^2}{ab}}$.

41. $\dfrac{\dfrac{a}{a+1}+\dfrac{1-a}{a}}{\dfrac{a}{a+1}-\dfrac{1-a}{a}}$.

42. $\dfrac{\dfrac{a-b}{a+b}+\dfrac{a+b}{a-b}}{1+\dfrac{2a^2b^2+2b^4}{a^4-b^4}}$.

43. $\left(\dfrac{x+y}{x-y}+2\right) \div \left(\dfrac{x+y}{x-y}-2\right)$.

44. $\left(a-\dfrac{1}{a^2}\right)^2\left(\dfrac{2}{a-1}\right) \div \left(a+1+\dfrac{1}{a}\right)$.

45. $\dfrac{\dfrac{2}{b}-\dfrac{1}{a+b}+\dfrac{1}{a-b}}{\dfrac{a+b}{a-b}-\dfrac{a-b}{a+b}}$.

46. $\dfrac{a^2+b^2}{a^2-b^2}-\dfrac{\dfrac{b}{a+b}}{1+\dfrac{2b-a}{a-b}}$.

47. $\dfrac{\left(x-\dfrac{y(x-y)}{x+y}\right)\left(x-\dfrac{y^2(x-y)}{x^2+y^2}\right)}{1-\dfrac{xy-y^2}{x^2}}$.

48. $\dfrac{2}{\dfrac{1}{b^2}\left(\dfrac{1}{c}-\dfrac{b}{2a}\right)}-b^2c\left(1-\dfrac{bc}{a-\dfrac{bc}{2}}\right)$.

49. $\dfrac{2b}{\dfrac{c}{2a}-2}-\dfrac{b}{ac}\left(\dfrac{4ab}{\dfrac{4b}{c}-\dfrac{b}{a}}-\dfrac{a}{\dfrac{1}{c}-\dfrac{1}{b}}\right)$.

50. If $x = \frac{a}{b+c}$, $y = \frac{b}{a+c}$, and $z = \frac{c}{a+b}$, find the value of $\frac{x}{x+1} + \frac{y}{y+1} + \frac{z}{z+1}$.

51. What value has $\frac{a^3 - b^3}{a^2 + b^2}$ when $a = x - \frac{1}{x}$ and $b = x + \frac{1}{x}$?

52. $1 - \cfrac{1}{1 - \cfrac{1}{1 + \cfrac{1}{2}}}$. **53.** $1 - \cfrac{1}{1 + \cfrac{2}{1 - \cfrac{2}{3}}}$. **54.** $1 + \cfrac{1}{1 - \cfrac{1}{1 + \cfrac{1}{a}}}$.

55. Brouncker (1620–1684) proved that π (the circumference of a circle divided by its diameter) is four times the following fraction:

$$\cfrac{1}{1 + \cfrac{1}{2 + \cfrac{9}{2 + \cfrac{25}{2 + \cfrac{49}{2 + \cfrac{81}{2 + \text{etc.}}}}}}}$$

(*a*) Rewrite the fraction, continuing it to $\frac{225}{2 + \text{etc.}}$

(*b*) Stopping with $2 + \frac{81}{2}$, find the difference in value between four times the value of this fraction and 3.1416, the approximate value of π.

NOTE. William Brouncker, one of the brilliant mathematicians of his time, was an intimate friend of John Wallis (see "First Course in Algebra," Revised Edition, p. 48). These two scientists were among the pioneers in the study of expressions with a countless number of terms.

The complex fraction in the exercise, if continued indefinitely according to the law which its form suggests, is called an infinite continued fraction. Brouncker was the first to study the properties of such expressions.

43. Equations involving fractions. Equations involving fractions are solved as in the

EXAMPLE

Solve $$5-\frac{8}{x}=\frac{5(x-1)}{x+1}. \qquad (1)$$

Solution. The L.C.M. of the denominators is $x(x+1)$.

$(1)\cdot x(x+1)$, $\quad 5x(x+1)-8(x+1)=5x^2-5x. \qquad (2)$

From (2), $\quad 5x^2+5x-8x-8=5x^2-5x, \qquad (3)$

or $\quad 2x=8.$

Whence $\quad x=4.$

Check. Substituting 4 for x in (1), $5-2=3$, or $3=3$.

For solving equations in one unknown which may or may not involve simple fractions we have the

Rule. *Where polynomial denominators occur, factor them if possible.*

Find the L.C.M. of the denominators of the fractions and multiply each fraction and each integral term of the equation by it, using cancellation wherever possible.

Transpose and solve as usual.

Reject all values for the unknown which do not satisfy the original equation.

EXERCISES

Solve the following for x and check as directed by the teacher:

1. $\frac{5x}{2}-2=2x.$
2. $\frac{7x}{3}-1=\frac{5x}{3}+3.$
3. $\frac{2x}{5}+1=x-\frac{7}{5}.$
4. $\frac{11x}{9}-\frac{2}{3}=x.$
5. $\frac{3x}{2}-\frac{16}{3}=x-\frac{25}{6}.$
6. $\frac{5x}{6}-\frac{21}{5}=\frac{5x-2}{10}.$
7. $\frac{2(x-4)}{3}-1=\frac{x+5}{3}.$
8. $2(x-2)=\frac{2+5x}{2}.$

9. $\frac{2x+1}{4} - \frac{1-2x}{3} = 9\frac{1}{4}$.

10. $\frac{3}{4x} + \frac{7}{16} = \frac{4}{3x}$.

11. $\frac{1}{3x} - \frac{2}{5x} = \frac{1}{45}$.

12. $3(x+2)-4(2x-3)+2=0$.

13. $\frac{x+5}{x-2} = \frac{10}{3}$.

14. $\frac{8}{7x+3} = \frac{3}{3x+1}$.

15. $\frac{2x-7}{3x-4} + \frac{3}{2} = 0$.

16. $\frac{3x-2}{x+7} + \frac{3}{5} = 0$.

17. $(x+5)(x+1)-(x-3)(x-2)=10$.

18. $\frac{3}{2x-5} + \frac{1}{2x-1} = 0$.

19. $\frac{2x-5}{2x+7} = \frac{3x-14}{3x-2}$.

20. $\frac{5x+3}{2x-3} = \frac{5x+7}{2x}$.

21. $\frac{1}{x} - \frac{1-x}{6-x^2} = 0$.

22. $\frac{x+1}{x+3} = \frac{x+4}{x+2}$.

23. $\frac{x+3}{2x+1} = \frac{3x+4}{6x-2}$.

24. $\frac{17-3x}{3x-13} = \frac{13-3x}{3x-8}$.

25. $\frac{x-3}{x+7} = 1 - \frac{5}{x+1}$.

26. $\frac{3(1-8x)}{5x} - \frac{2(1+8x)}{8x-1} = \frac{1-34x}{5x}$.

27. $\frac{3}{2x+1} + \frac{2x+1}{1-2x} = 1 - \frac{8x^2}{4x^2-1}$.

28. $1 - \frac{c+x}{x+1} = \frac{x(c-1)}{c(x+1)}$.

29. $\frac{1+2x}{2x-1} - \frac{3x+1}{x+1} = \frac{8+3x-4x^2}{2x^2+x-1}$.

30. $\frac{x}{a} - a = \frac{x}{b} - b$.

31. $\frac{x+a}{a} - \frac{x+b}{b} = b - a$.

32. $\frac{1}{a} + \frac{1}{b} = \frac{1}{x}$.

33. $\frac{x}{a-1} + \frac{x}{a+1} = 2a$.

34. $\frac{x}{a+2} - \frac{x+16}{a-2} = 4a$.

35. $\dfrac{2x+1}{x+3}+\dfrac{3x-7}{2-x}=\dfrac{9-3x-x^2}{x^2+x-6}$.

36. $\dfrac{x+cd}{c+d}-\dfrac{x-cd}{c-d}=\dfrac{2c^2d-2cd^2}{c^2-d^2}$.

37. $\dfrac{x-a}{2x-a}-\dfrac{3x-c}{6x-c}=0$.

38. $\dfrac{5x+.4}{.6}+\dfrac{1.3x-.05}{4}=\dfrac{30.35-8x}{2.4}$.

39. $11.3-\dfrac{5-2x}{.5}=2.3-(5-7x)+\dfrac{x+2}{.2}$.

40. $\dfrac{3x-1}{.25}+\dfrac{x-4}{.5}=3(3x-14)$.

41. $\dfrac{2x-.3}{2x}-\dfrac{.4x+5}{x}=\dfrac{275}{.25x}-9.4$.

PROBLEMS

1. Separate the number 286 into two parts such that the greater will be $2\frac{1}{7}$ times the less.

2. Separate the number 1010 into three parts such that the second will be $\frac{11}{6}$ of the first and the third will be $\frac{25}{9}$ of the first.

3. By what number must 352 be divided so as to give a partial quotient 15 and the remainder 7?

4. What number must be subtracted from both terms of the fraction $\frac{32}{39}$ to give a fraction equivalent to $\frac{2}{3}$?

5. Separate 133 into two parts such that their quotient is $2\frac{1}{6}$.

6. Separate 96 into two parts such that 56 exceeds two thirds of the one by as much as the other exceeds 16.

7. A boy is 12 years old and his sister 8 years old. In how many years will the boy be $\frac{6}{5}$ as old as his sister?

8. Two thirds a man's age now equals $\frac{3}{2}$ his age 30 years ago What is his age?

9. The square of a certain number is 4 greater than two thirds of the product of the next two consecutive numbers. Find the number.

10. The length of a certain rectangle is $2\frac{1}{2}$ times the width. If it were 10 yards shorter and $1\frac{1}{2}$ yards wider, its area would be 1260 square feet less. Find the dimensions of the rectangle.

11. A square court has $\frac{6}{5}$ the area of a rectangular court whose length is 4 yards greater and whose width is 3 yards less. Find the dimensions of the square court.

12. A can do a piece of work in 10 days and B in 12 days. How many days will they both require working together?

HINT. Let x = number of days required by A and B together. Then $\frac{1}{x}$ = fractional part of the piece of work that they can do in 1 day.

13. A can do a piece of work in 10 days and B in 15 days. After they have worked together 5 days, how many days will A require to finish the work?

14. A tank has a supply pipe which fills it in 4 hours and a waste pipe which empties it in 6 hours. If the tank is empty and both pipes are opened, how much time must elapse before the tank is filled?

15. A tank has a supply pipe which fills it in 4 hours and two waste pipes which empty it in 6 and 8 hours respectively. If the tank is full and all three pipes are open, how much time will be required to empty the tank?

16. If all three pipes of Problem 15 were outlet pipes, how long would be required to empty the tank?

17. If in Problem 15 the supply pipe had been closed after 4 hours, how much more time would have been required to empty the tank?

18. The diameter of the earth is $3\frac{2}{3}$ times that of the moon, and the difference of the two diameters is 5760 miles. Find each diameter in miles.

19. The diameter of the sun is 3220 miles greater than 109 times the diameter of the earth, and the sum of the two diameters is 874,420 miles. Find each diameter in miles.

20. The diameter of Jupiter is $10\frac{10}{11}$ times the diameter of the earth, and the sum of their diameters is 94,320 miles. Find each diameter in miles.

21. A man who can row 4 miles per hour in still water rows up a stream the rate of whose current is $1\frac{1}{2}$ miles per hour. After rowing back he finds that the entire journey required 10 hours. Find the time required for the trip upstream.

22. A man who can row $4\frac{1}{2}$ miles per hour in still water finds that it requires $6\frac{1}{6}$ hours to row upstream a distance which it requires $2\frac{5}{6}$ hours to row down. Find the rate of the current.

23. A passenger train whose rate is 40 miles per hour leaves a certain station 2 hours and 45 minutes after a freight train. The passenger train overtakes the freight in 5 hours and 15 minutes. Find the rate of the freight train in miles per hour.

24. A man invests a part of $8000 at 5% and the remainder at 4%. If the yearly interest on the whole investment is $345, how much was invested at each rate?

25. A man invests $6800 in two parts: the first part at 5%, and the second part at 4%. If the average rate of interest is $4\frac{3}{4}$%, find the amount of each investment.

26. Two thousand dollars of Mr. A's income is not taxed. All of his income over that amount is taxed 2%, and all above $10,000 is taxed 2% in addition. He pays a tax of $180. What is his income?

27. How much water must be added to a gallon of alcohol 95% pure so as to make a mixture 10% pure?

HINT. Let w = the number of gallons of water to be added.

Then $$\frac{\frac{95}{100}\cdot 1}{1+w}=\frac{10}{100},\text{ etc.}$$

44. If $3m + 5$ packages weigh p pounds, what is the weight of n of them?

45. If it takes a man t hours to do a piece of work, what portion of the work can he do in 1 hour? What portion of the work would n men do in 1 hour? What portion would k men do in h hours?

46. If it takes n men h hours to do a piece of work, how long will it take x men to do it?

47. A man buys bananas at d cents a dozen and sells them for b cents each. What does he gain on n dozen?

48. A man bought m articles for c cents per hundred. He sold them all for $10. How many dollars did he lose?

49. If n yards of ribbon cost c cents, find the cost of x yards.

50. If y yards of ribbon cost x cents, how many yards can be bought for d dollars?

51. A man buys goods for b dollars and sells them for s dollars. What is his per cent of gain?

52. One man can do a piece of work in d days, another can do the same work in n days. How many days will it take both, working together?

53. If it takes h hours to mow m acres, how many days of 10 hours each will it take to mow n acres?

54. A train goes y yards in t seconds. If this equals m miles per hour, write an equation involving y, t, and m.

55. If n men can do a piece of work in d days, how many men would it be necessary to hire if the work had to be done in t days?

56. A transport plying between two ports is under fire for f feet of the way. If she steams k knots per hour, for how many minutes is she under fire?

HINT. 1 knot = 6080 feet.

CHAPTER V

LINEAR SYSTEMS

44. Graphical solution of a linear system. The construction of the graph of a single linear equation in two unknowns or of a linear system in two unknowns depends on several assumptions and definitions. It is agreed:

I. To have at right angles to each other two lines, $X'OX$, called the **x-axis**, and $Y'OY$, called the **y-axis**.

II. To have a line of definite length for a unit of distance.

Thus the number 2 will correspond to a distance of twice the unit, the number $4\frac{1}{2}$ to a distance $4\frac{1}{2}$ times the unit, etc.

III. That the distance (measured parallel to the x-axis) from the y-axis to any point in the paper be the **x-distance** (or abscissa) of the point, and the distance (measured parallel to the y-axis) from the x-axis to the point be the **y-distance** (or ordinate) of the point.

IV. That the x-distance of a point to the *right* of the y-axis be represented by a *positive* number, and the x-distance of a point to the *left* by a *negative* number; also that the y-distance of a point *above* the x-axis be represented by a *positive* number, and the y-distance of a point *below* the x-axis by a negative number. Briefly, *distances measured from the axis to the **right** or **upward** are **positive**; to the **left** or **downward** are **negative**.*

V. That every point in the surface of the paper corresponds to a *pair of numbers*, one or both of which may be positive, negative, integral, or fractional.

VI. That of a given pair of numbers the first be the measure of the x-distance and the second the measure of the y-distance.

Thus the point (2, 3) is the point whose x-distance is 2 and whose y-distance is 3.

The point of intersection of the axes is called the **origin.**

The values of the x-distance and the y-distance are often called the **coördinates** of the point.

The relation between an equation and its graph may be stated as follows:

The equation of a line is satisfied by the values of the x-distances and the y-distance of any point on that line.

Any point, the values of whose x-distance and whose y-distance satisfy the equation, is on the graph of the equation.

The graph of a linear equation in two unknowns is a straight line. Therefore it is necessary in constructing the graph of such an equation to locate only two points whose coördinates satisfy the equation and then to draw through the two points a straight line. It is usually most convenient to locate the two points where the line cuts the axes. If these two points are very close together, however, the direction of the line will not be accurately determined. This error can be avoided by selecting two points at a greater distance apart.

The *graphical solution* of a linear system in two unknowns consists in plotting the two equations to the same scale and on the same axes and obtaining from the graph the values of x and y at the point of intersection of the lines.

Through the graphical study of equations we unite the two subjects of geometry and algebra, which have hitherto seemed quite separate, and learn to interpret problems of the one in the language of the other.

EXAMPLE

Solve graphically the system

$$3x - 4y + 20 = 0,$$
$$2x + y + 6 = 0.$$

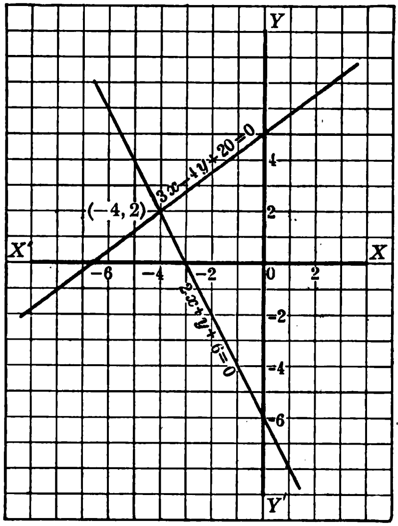
Solution. Substituting zero for x in $3x - 4y + 20 = 0$, we obtain $y = 5$. Substituting zero for y, we obtain $x = -6\frac{2}{3}$. This may be expressed in tabular form:

If $x =$	0	$-6\frac{2}{3}$
then $y =$	5	0

Similarly, for the equation $2x + y + 6 = 0$ we obtain the following table:

If $x =$	0	-3
then $y =$	-6	0

Then, constructing the graph of each equation as indicated in the adjacent figure, we obtain for the coördinates of the point of intersection of the two lines $x = -4$ and $y = 2$.



EXERCISES

Solve graphically:

1. $2x + y = 8,$
   $x + 2y = 13.$

2. $x - y = 6,$
   $5x + 4y = -15.$

3. $x + 2y + 11 = 0,$
   $y - x = 7.$

4. $x + 2y = 4,$
   $6y + 2x = 20.$

5. $y + 4 = 10x,$
   $2 - x = 0.$

6. $2x + 4y = 20,$
   $2y - 2 = x.$

7. $x + y = 5,$
$y + 2 = 0.$

8. $x + 5 = -3y,$
$6y + 2x - 11 = 0.$

9. $x + y = 4,$
$x + 2y = 7.$

10. $x - y = 5,$
$3x + 2y = 5.$

11. $3x - 4y = 33,$
$4x + 3y = -6.$

12. $2x - 4y = 9,$
$x - 2y = 8.$

13. In each of the first three exercises will the values of the x- and y-distances of the point of intersection of the two lines, as obtained from the graph, satisfy the equation obtained by adding the two given equations?

14. Graph the equation $x - 2y = 8$. Then multiply both members by 3 and graph the resulting equation. Compare the two graphs. Then use -2 as a multiplier and graph the resulting equation. Compare the three graphs. What conclusion seems warranted?

15. What are the coördinates of the origin?

16. Is a graphical solution of a linear system ever impossible? Give an example.

17. What is the form of the equation of a line parallel to the x-axis? the y-axis?

18. The boiling point of water on a centigrade thermometer is marked 100°, and on a Fahrenheit 212°. The freezing point on the centigrade is zero and on the Fahrenheit 32°. Consequently a degree on one is not equal to a degree on the other, nor does a temperature of 60° Fahrenheit mean 60° centigrade. Show that the correct relation is expressed by the equation $C = \frac{5}{9}(F - 32)$, where C represents the number of degrees centigrade and F the number of degrees Fahrenheit.

19. Construct a graph of the equation in Exercise 18, using C and F as abscissa and ordinate. Can you, by means of this graph, express a centigrade reading in degrees Fahrenheit and vice versa?

20. By means of the graph drawn in Exercise 19 express the following centigrade readings in Fahrenheit readings and vice versa: (*a*) 60°C.; (*b*) 150°F.; (*c*) – 20°C.; (*d*) – 30°F.

21. From the graph determine what reading means the same temperature on both scales.

45. Elimination. In order to find values of x and y which satisfy the equation

$$3x + 2y = 20, \tag{1}$$

when we know that

$$y = 2x + 3, \tag{2}$$

we may substitute for y in the first equation the value of y from the second, obtaining the single equation in x,

$$3x + 2(2x + 3) = 20, \quad \text{or} \quad 7x = 14.$$

The process by which we have obtained one equation containing one unknown from the two equations (1) and (2) each of which contains two unknowns illustrates one method of elimination.

In general, the process of deriving from a system of n equations a system of $n-1$ equations, containing one variable less than the original system, is called **elimination.**

For example, when $n = 2$, if we have a system of two equations in two unknowns, the process of elimination leads to one equation in one unknown. Since we can always solve such an equation, it appears that we can solve a system of two equations in two unknowns whenever it is possible to eliminate one of the unknowns. We shall see that only in certain exceptional cases is elimination impossible. This is either because more than one unknown is removed when we try it or because the result of attempted elimination is not an equation.

Only two methods of solution will be considered — that involving **elimination by substitution** and that involving **elimination by addition or subtraction.**

RE

46. Solution by substitution. The method of solving a system of two linear equations by substitution is illustrated in the

EXAMPLE

Solve the system $\begin{cases} x - 13y = 31, & (1) \\ 8x + 11y = 18. & (2) \end{cases}$

Solution. From (1), $$x = 13y + 31. \quad (3)$$

Substituting this value for x in (2),

$$8(13y + 31) + 11y = 18.$$

Simplifying, $104y + 248 + 11y = 18.$

Combining terms, $$115y = -230.$$

Whence $$y = -2.$$

Substituting -2 for y in (3), and solving,

$$x = 5.$$

Check. Substituting 5 for x and -2 for y in (1) and (2),

$$5 - 13(-2) = 31, \text{ or } 31 = 31,$$

and $$8 \cdot 5 + 11(-2) = 18, \text{ or } 18 = 18.$$

The method of the preceding solution is stated in the following

Rule. *Solve either equation for the value of one unknown in terms of the other.*

Substitute this value for the unknown which it represents, in the equation from which it was not obtained, and solve the resulting equation.

In the simplest of the preceding equations which contains both unknowns, substitute the definite value just found, and solve, thus obtaining a definite value for the other unknown.

Check. *Substitute for each unknown in both original equations its value as found. If the resulting equations are not obvious identities, simplify them until they become so.*

EXERCISES

Solve by substitution:

1. $3x - 8y = 20,$
   $x - 6y = 0.$

2. $2x + 5y = 8,$
   $x - 10y = 9.$

3. $2(x + y) + 3y = 4,$
   $5 = x + y.$

4. $16x + 7 = 15y,$
   $4x + 5y = 0.$

5. $3x - 2y = 18,$
   $4y + 3x = 0.$

6. $\dfrac{8m - 3n}{2} + 6n = -9,$
   $4m - 1 = 3n.$

7. $4t - 2n = 18,$
   $20t = 7n + 63.$

8. $6x + 38 = 12y,$
   $4x - 8y = 0.$

47. Solution by addition or subtraction. The method of solving a system of two linear equations by addition or subtraction is illustrated in the

EXAMPLE

$$\text{Solve the system} \begin{cases} 13x + 3y = 14, & (1) \\ 7x - 2y = 22. & (2) \end{cases}$$

Solution. First eliminate y, as follows:

$$\begin{array}{lrl} (1)\cdot 2, & 26x + 6y = 28 & (3) \\ (2)\cdot 3, & \underline{21x - 6y = 66} & (4) \\ (3)+(4), & 47x \qquad = 94 & (5) \\ (5) \div 47, & x = 2. & (6) \end{array}$$

Substituting 2 for x in (2), we obtain $y = -4$.

Check. Substituting 2 for x and -4 for y in (1) and (2),

$$26 - 12 = 14, \text{ or } 14 = 14,$$

and

$$14 + 8 = 22, \text{ or } 22 = 22.$$

Before starting to eliminate, the system should always be inspected carefully to determine which unknown can be removed most conveniently. In this case the elimination of y is the simpler, because it involves multiplication by smaller numbers than does the elimination of x.

The method of the preceding solution is stated in the following

Rule. *If necessary, multiply each member of the first equation by a number, and each member of the second equation by another number, such that the coefficients of the same unknown in the resulting equations will be numerically equal.*

If these coefficients have like signs, subtract one equation from the other; if they have unlike signs, add; then solve the equation thus obtained.

In the simplest of the preceding equations which contains both unknowns, substitute the value just found and solve for the other unknown.

Check. As on page 74.

ORAL EXERCISES

Solve the following systems:

1. $x + y = 4,$ $x - y = 2.$
2. $x + 2y = 3,$ $x + y = 2.$
3. $x - y = 5,$ $2x + y = 4.$
4. $2x + 3y = 8,$ $x - y = 1.$
5. $x + 3y = 0,$ $2x - 3y = 9.$
6. $5x - 6y = 7,$ $4x - 3y = 2.$

48. Special cases. The equation $x + y = 10$ has as roots any set of two numbers whose sum is 10. If $x + y = 5$ is taken as the other equation of a system, one can see immediately that the two equations have no set of roots in common, since the sum of two numbers cannot be 10 and 5 at the same time.

A system of equations which has a common set of roots is called a **simultaneous** system.

A system of equations which does not have a common set of roots is called **inconsistent** or **incompatible**.

The attempt to solve an incompatible system results in getting rid not only of one but of both unknowns and leads to a statement in the form of an equation which is false.

Consider $$x + y = 10, \qquad (1)$$

$$x + y = 5. \qquad (2)$$

(1) − (2), $$0 = 5, \text{ which is false.}$$

If, on the other hand, the equation $x + y = 10$ is taken for one equation of a system and $2x + 2y = 20$ for the other, it appears that any set of numbers which satisfies one equation satisfies also the other, since if the sum of two numbers is 10, the sum of twice those numbers is 20, and any one of the countless sets of roots of one equation is a set of roots of the other. In fact, the second equation may be obtained from the first by multiplying each member by 2.

If one equation of a system can be obtained from one or more of the other equations of the system by application of one or more of the axioms, it is called a **derived** or **dependent** equation. If it cannot be so obtained, it is called **independent.**

Thus equations (1) and (2) in the example on page 75 are independent, while equation (5) is derived from them.

An attempt to eliminate one unknown from a system of two equations in two unknowns which are not independent results in getting rid not only of both unknowns but of the constant terms as well, so that only the identity $0 = 0$ remains.

Thus $$x + y = 10, \qquad (1)$$

$$2x + 2y = 20. \qquad (2)$$

(1) · 2, $$2x + 2y = 20. \qquad (3)$$

(2) − (3), $$0 = 0.$$

ORAL EXERCISES

Which of the following systems are incompatible, which are simultaneous, and which are dependent?

1. $x + y = 2,$
   $x + y = 7.$

2. $3x + 3y = 6,$
   $x + y = 2.$

3. $2x + y = 4,$
   $4x + 2y = 1.$

4. $x + y = 3,$
   $x - y = 1.$

5. $2x + 3y = 4,$
   $4x + 6y = 8.$

6. $x - 2y = 4,$
   $5x - 10y = 4.$

7. $3x - y = 2,$
   $9x - 3y = 6.$

8. $3x - y = 2,$
   $9x + 3y = 6.$

9. $2y - x + 6 = 0,$
   $3x - 6y - 21 = 0.$

10. $2x = 7y - 5,$
    $8x = 28y + 3.$

EXERCISES

Solve the following systems:

1. $\dfrac{2x}{3} + y = -7,$
   $5x - 3y = 105.$

2. $\dfrac{3r}{4} - \dfrac{7}{2} = \dfrac{s}{12},$
   $r + 8 = -2s.$

3. $\dfrac{3y+1}{4} - \dfrac{z+22}{12} = 3,$
   $z - 2y = 1.$

4. $.5x + .7y = \frac{3}{5},$
   $.8x - .2y = 3\frac{3}{5}.$

5. $\dfrac{m+2n}{5} - \dfrac{2m-n}{10} = \dfrac{5}{2},$
   $\dfrac{m+n}{4} - \dfrac{m-n}{7} = \dfrac{n}{2}.$

6. $\dfrac{1}{x} + \dfrac{1}{y} = -\dfrac{1}{6},$
   $\dfrac{2}{x} - \dfrac{3}{y} = \dfrac{4}{3}.$

   HINT: Solve first for $\frac{1}{x}$ and $\frac{1}{y}$.

7. $\dfrac{2r+4s}{2r-s} = \dfrac{38}{3},$
   $\dfrac{5}{r} - \dfrac{7}{s} = 0.$

8. $\dfrac{2m+3n-2}{m+n+6} = \dfrac{4}{3},$
   $\dfrac{1}{m} + \dfrac{1}{n} = \dfrac{6}{m}.$

9. $\dfrac{3}{x-3}-\dfrac{1}{y}=0,$
$\dfrac{x-6y}{5}=\dfrac{2}{5}.$

10. $\dfrac{2.5h+1-.1k}{k-10+h}=2.5,$
$\dfrac{1}{.8h-2.2}+\dfrac{20}{35-5.5k}=0.$

11. $\dfrac{\dfrac{2h-3k}{4}}{k-h}=3,$
$\dfrac{4h+5k}{\dfrac{2}{3}}=39.$

12. $\dfrac{\dfrac{3(x+y)}{3}}{5}+\dfrac{\dfrac{x-y}{-11}}{5}=0,$
$2x+y=7.$

13. $\dfrac{\dfrac{x}{5}+\dfrac{y}{2}}{\dfrac{3}{4}}=\dfrac{\dfrac{5y}{2}+\dfrac{2x}{5}}{\dfrac{3}{2}},$
$\dfrac{x}{4-2y}=\dfrac{1}{4}.$

14. $\dfrac{3x+z}{z-x}+2=\dfrac{12}{x-z},$
$\dfrac{x+7}{z}+\dfrac{17+x}{5-z}=0.$

Solve for x and y:

15. $5x+4y=10a+4,$
$x-2ay=0.$

16. $7x+5y=21c,$
$\dfrac{x}{5}+\dfrac{y}{7}=3c.$

17. $2ax-3by=7,$
$5ax+7by=3.$

18. $\dfrac{2cx}{a}-\dfrac{y}{a}=5c,$
$\dfrac{2x}{3}-\dfrac{y}{c}=a.$

19. $\dfrac{x}{a}-\dfrac{y}{c}=6,$
$\dfrac{x}{2a}+\dfrac{y}{3c}=13.$

20. $\dfrac{a}{x}+\dfrac{3a}{y}=1,$
$\dfrac{3a}{x}+\dfrac{a}{y}=\dfrac{1}{2}.$

21. $\dfrac{x+y}{4a}=3+\dfrac{x-y}{7},$
$\dfrac{2x-y}{a}=6.$

22. $\dfrac{x}{a}+\dfrac{y}{b}=\dfrac{a+b}{ab},$
$x-y=\dfrac{a^2-b^2}{ab}.$

23. $2ax-4by=7c,$
$3ax-6by=5c.$

24. $\begin{aligned} ax + by &= c, \\ kax + kby &= ck. \end{aligned}$

25. $\begin{aligned} ax + by &= c, \\ dx + fy &= g. \end{aligned}$

26. Show that if $af - bd = 0$, the equations in Exercise 25 are inconsistent, unless they form a dependent system.

27. Solve the system of Exercise 17 for a and b in terms of x and y.

28. Solve the system of Exercise 23 for a and b in terms of x, y, and c.

49. Equations in several unknowns. We have already seen that the equation $x + y = 10$ is satisfied by an unlimited number of sets of roots, since there is an infinity of pairs of numbers whose sum is 10.

An equation or a system of equations which is satisfied by an infinite number of sets of roots is said to be **indeterminate.**

If a simultaneous system is satisfied by only a limited number of sets of roots, it is said to be **determinate.**

The system $x + y = 10, x - y = 2$, is determinate and has the set of roots (6, 4). The system $2x + 2y = 20$, $x + y = 10$, is indeterminate.

When we consider systems of equations in three unknowns, the question arises whether two such equations form a determinate system. For example, the equation

$$x + y + z = 10 \tag{1}$$

is satisfied by an infinite number of sets of roots. If we consider a system consisting of (1) and

$$x - (y + z) = 2, \tag{2}$$

it appears from inspection that the system is satisfied if $x = 6$ and $y + z = 4$. But the equation $y + z = 4$ is satisfied by an infinite number of sets of roots. Hence equations (1) and (2) form an indeterminate system.

If, however, we adjoin a third equation to the system, as

$$y - z = 2, \tag{3}$$

it becomes determinate, since $y + z = 4$ and $y - z = 2$ are satisfied only by the set of numbers 3, 1. It is usually true that three equations in three unknowns form a determinate system.

In general, when the number of unknowns in a system of linear equations exceeds the number of equations, the system is indeterminate. If the number of equations equals the number of unknowns, the system is usually determinate and simultaneous. If the number of equations exceeds the number of unknowns, the system is usually inconsistent. There are many special cases which arise in the study of linear systems in n unknowns, corresponding to those mentioned for two unknowns in section 48, but they become very complicated for larger values of n, and a thorough study of them is quite beyond the scope of this text.

NOTE. It is not a little remarkable that the writings of the first great algebraist, Diophantos of Alexandria (about A.D. 275), are devoted almost entirely to the solution of indeterminate equations; that is, to finding the sets of related values which satisfy an equation in two unknowns, or, perhaps, two equations in three unknowns. We know practically nothing of Diophantos himself, except the information contained in his epitaph, which reads as follows: "Diophantos passed one sixth of his life in childhood, one twelfth in youth, one seventh more as a bachelor; five years after his marriage a son was born who died four years before his father, at half his father's age." From this statement the reader was supposed to be able to find at what age Diophantos died. As a mathematician Diophantos stood alone, without any prominent forerunner or disciple, so far as we know. His solutions of the indeterminate equations were exceedingly skillful, but his methods were so obscure that his work had comparatively little influence upon later mathematicians.

50. Determinate systems. The method of obtaining the set of roots of a determinate system in three unknowns is illustrated in the

EXAMPLE

Solve the system $\begin{cases} x + 6y - 5z = 21, & (1) \\ 3x - 8y + z = -5, & (2) \\ 5x - 7y + 2z = 4. & (3) \end{cases}$

Solution. First eliminate one unknown, say z, between (1) and (2):

$$x + 6y - 5z = 21. \qquad (1)$$

(2) · 5, $$15x - 40y + 5z = -25. \qquad (4)$$

(1) + (4), $$16x - 34y = -4. \qquad (5)$$

Now eliminate z between (2) and (3):

(2) · 2, $$6x - 16y + 2z = -10. \qquad (6)$$

$$5x - 7y + 2z = 4. \qquad (3)$$

(6) − (3), $$x - 9y = -14. \qquad (7)$$

The equations (5) and (7) contain the same two unknowns x and y.

$$16x - 34y = -4. \qquad (5)$$

(7) · 16, $$16x - 144y = -224. \qquad (8)$$

(5) − (8), $$110y = 220.$$

$$y = 2.$$

Substituting in (7), $$x = 4.$$

Substituting both these values in (1),

$$4 + 12 - 5z = 21.$$

Whence $$z = -1.$$

Check. Substituting 4 for x, 2 for y, and -1 for z in (1), (2), and (3) respectively,

$$4 + 6 \cdot 2 - 5(-1) = 21, \quad \text{or} \quad 21 = 21.$$

$$3 \cdot 4 - 8 \cdot 2 + (-1) = -5, \quad \text{or} \quad -5 = -5.$$

$$5 \cdot 4 - 7 \cdot 2 + 2(-1) = 4, \quad \text{or} \quad 4 = 4.$$

For the solution of a simultaneous system of equations in three unknowns we have the

Rule. *From an inspection of the coefficients decide which unknown is most easily eliminated.*

Using any two equations, eliminate that unknown.

With one of the equations just used and the third equation again eliminate the same unknown.

The last two operations give two equations in the same two unknowns. Solve these equations.

Substitute in the simplest of the original equations the two values found, and solve for the third unknown.

Check. *Substitute the values found in the original equations and simplify results.*

A system of four independent equations in four unknowns may be solved as follows:

Use the first and second equation, then the first and third, and lastly the first and fourth, and eliminate the same unknown each time. This gives a system of three equations in the same three unknowns, which can be solved by the rule given above.

EXERCISES

Solve for x, y, and z and check the results:

1. $$\begin{aligned} x + 3y - 5z &= 2, \\ 2x - y - z &= 1, \\ 3x + 5y - 7z &= -10. \end{aligned}$$

2. $$\begin{aligned} 2x + 3y + 4z &= -14, \\ x - y + 3z &= 0, \\ 5x + 2y + z &= 14. \end{aligned}$$

3. $$\begin{aligned} x + 2y + z &= -1, \\ 2x - y + z &= -20, \\ -x - y - 5z &= 13. \end{aligned}$$

4. $$\begin{aligned} x + y + z &= 1, \\ x + y - z &= 2, \\ x - y + z &= 3. \end{aligned}$$

5. $$\begin{aligned} x + 2y + z &= 1, \\ 2x + y - z &= 0, \\ x + 2y + z &= 0. \end{aligned}$$

6. $$\begin{aligned} 2x - y + 5z &= 0, \\ 8x + 7y + z &= 38, \\ x - 5y - z &= 7. \end{aligned}$$

7. $\begin{aligned} 2x + 3y + 6z &= 6,\\ 4x + y + z &= 3,\\ x - 6y + 3z &= \tfrac{1}{2}. \end{aligned}$

8. $\begin{aligned} 4x - 3y &= z,\\ z &= x + y,\\ 2x &= 3y + 1. \end{aligned}$

9. $\begin{aligned} 2x + y &= 5 + z,\\ x - 2z &= 6,\\ 3y + 2z &= x. \end{aligned}$

10. $\begin{aligned} x - 2y &= 10,\\ 3y + 4z &= -1,\\ 5x - z &= 18. \end{aligned}$

11. $\begin{aligned} x &= 3z + 2,\\ y &= x - 7\tfrac{1}{2},\\ z &= 6y - 1. \end{aligned}$

12. $\begin{aligned} 4x - 2y &= 0,\\ 6z - 8y &= -2,\\ x + z &= 1\tfrac{1}{4}. \end{aligned}$

13. $\begin{aligned} x + y &= 3a,\\ x + z &= 4a,\\ y + z &= 5a. \end{aligned}$

14. $\begin{aligned} 3x + 2y &= 6a - 2z,\\ x - 5z + 6y &= 2a - 11b,\\ 6x - 8y &= 12a + 8b. \end{aligned}$

15. $\begin{aligned} ax + by &= 0,\\ cx - bz &= 2bc,\\ bx + az - cy &= b^2. \end{aligned}$

16. $\begin{aligned} .3x + .2y + .4z &= 1.9,\\ .02x &= .1 - .01y - .02z,\\ x + y + z &= 6. \end{aligned}$

17. $\begin{aligned} \frac{a}{x} + \frac{b}{y} + \frac{c}{z} &= 3,\\ \frac{a}{x} - \frac{b}{y} + \frac{2c}{z} &= 2,\\ \frac{5a}{x} - \frac{2c}{z} &= \frac{3b}{y}. \end{aligned}$

18. $\begin{aligned} \frac{x+y}{3} - \frac{z}{5} &= \frac{2}{3},\\ \frac{y+z}{4} - \frac{x}{2} &= \frac{5}{12},\\ 3x + 2y &= 12 - 3z. \end{aligned}$

19. $\begin{aligned} hx + ky - lz &= 2hk,\\ ky - hx + lz &= 2kl,\\ hx - ky + lz &= 2hl. \end{aligned}$

20. $\begin{aligned} 2x + y + z + w &= 1,\\ x - y - z + 2w &= 4,\\ x + 2y - z - w &= 0,\\ x - y + 2z - w &= 1. \end{aligned}$

21. $\begin{aligned} 2x - y + z - w &= 2,\\ 3x + y - z - w &= 0,\\ 4x - 2y + z - w &= -9,\\ 2x - 3y - 2z + w &= 7. \end{aligned}$

22. $\begin{aligned} x + y - z &= 5,\\ x + y - w &= 2,\\ x - y + z + w &= 5,\\ 2x - 3y - z + w &= 7. \end{aligned}$

23. $\begin{aligned} x + y + z &= 1,\\ x - y - w &= -1,\\ x - z - w &= -5,\\ y - z + w &= 0. \end{aligned}$

ROBLEMS

the following problems by means
, and check the results:

s is 109 and their difference

days. He is paid $3.50 per
80 cents for each day that
end of the 30 days he re-
receive board?

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work can be done in a
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10. A sum of $4000 is invested, a part in 5% bonds at 90, and the remainder in 6% bonds at 110. If the total annual income is $220, find the sum invested at each rate.

11. The purity of gold is measured in carats, 18 carats meaning that 18 parts out of 24 are pure gold. A goldsmith has 20 ounces of pure gold which he wishes to use in making 16-carat and 10-carat alloys. How much pure gold can he use for each alloy if he makes 39 ounces in all?

12. A bag weighing 18 ounces contains two sizes of steel balls—ounce balls and $\frac{3}{8}$-ounce balls. There are 23 balls in all. Find the number of balls of each size.

13. If the length of a rectangle is decreased by 7 feet and the breadth is increased by 8 feet, the area is unchanged. If the length is increased by 14 feet and the breadth is decreased by 4 feet, the area is also unchanged. Find the dimensions of the rectangle.

14. A man has $4.50 in dimes and quarters. If he has 36 coins in all, how many has he of each?

15. A man has $6.00 in quarters, dimes, and nickels. He has as many quarters as he has dimes, and three times as many nickels as dimes. How many of each has he?

16. A is half as old as B. Seven years ago A was one third as old as B. How old is each now?

17. A man can walk 4 miles per hour. He reaches a point 20 miles from his starting point in three hours, having been taken part of the way by a stage traveling 12 miles per hour. How far did the stage carry him?

18. A man and his two sons can do a piece of work in $\frac{4}{5}$ of a day. The two boys together can do it in $1\frac{5}{7}$ days and one of them can do it in 1 day less than the other. What portion of the work does each do when they work together?

19. Two automobiles 25 miles apart travel toward each other and meet in 1 hour. If they had both traveled in the same direction, the faster would have overtaken the slower in 5 hours. Find the rate of each.

20. An aëroplane travels a certain distance in 3 hours. If the distance had been half again as great, the aëroplane would have been forced to travel 50 miles per hour faster in order to cover it in the same time. Find the distance and the speed of the aëroplane.

21. One angle of a triangle is twice another, and their sum equals the third. Find the number of degrees in each angle of the triangle.

22. The sum of three numbers is 108. The sum of one third the first, one fourth the second, and one sixth the third is 25. Three times the first added to four times the second and six times the third is 504. Find the numbers.

23. The sum of three numbers is 217. The quotient of the first by the second is 5, which is also the quotient of the second by the third. Find the numbers.

24. If the tens' and units' digits of a three-digit number be interchanged, the resulting number is 27 less than the given number. If the same interchange is made with the tens' and hundreds' digits, the resulting number is 180 less than the given number. The sum of the digits is 14. Find the number.

25. In 1 hour a tank which has three intake pipes is filled seven-eighths full by all three together. The tank is filled in $1\frac{1}{3}$ hours if the first and second pipes are open, and in 2 hours and 40 minutes if the second and third pipes are open. Find the time in hours required by each pipe to fill the tank.

26. The sum of two sides which meet at one of the vertices of a quadrilateral is 20 feet. The sum of the two which meet at the next vertex is 27 feet. The sums of the two pairs of

opposite sides are 23 feet and 29 feet respectively. Find each side. (Two solutions.)

27. Two chairs cost h dollars. The first cost m cents more than the second. Find the cost of each in cents.

28. Find two numbers whose sum is a and whose difference is b.

29. Find two numbers whose sum is $a + b$ and whose difference is $a - b$.

30. Two relays of messengers carry a message k miles. The first relay travels c miles further than the second. How far does each go?

31. A man has a dollars and b cents in dimes and quarters. If he has c coins in all, how many of each kind has he?

32. A man has a dollars in quarters and nickels, with b more quarters than nickels. How many of each has he?

33. A and B together can do a piece of work in m days. B works c times as fast as A. How many days does each require to do the work alone?

34. A man rows m miles downstream in t hours and returns in a hours. Find his rate in still water and the rate of the river.

35. Two contestants run over a 440-yard course. The first wins by 4 seconds when given a start of 200 feet. They finish together when the first is given a handicap of 40 yards. Find the rate of each in feet per second.

36. A train leaves M two hours late and runs from M to P at 50% more than its usual rate, arriving on time. If it had run from M to P at 25 miles per hour, it would have been 48 minutes late. Find the usual rate and the distance from M to P.

37. A train leaves M 30 minutes late. It then runs to N at a rate 20% greater than its usual rate, arriving 6 minutes late. Had it run 15 miles of the distance from M to N at the usual

rate and the rest of the trip at the increased rate, it would have been 12 minutes late. Find the distance from M to N and the usual rate of the train.

38. It is desired to have a 10-gallon mixture of 45% alcohol. Two mixtures, one of 95% alcohol and another of 15% alcohol, are to be used. How many gallons of each will be required to make the desired mixture?

HINT. Let x and y = the number of gallons of 95% and 15% alcohol respectively. Then $\frac{.95x + .15y}{10} = \frac{45}{100}$, and $x + y = 10$.

39. A chemist has the same acid in two strengths. Eight liters of one mixed with 12 liters of the other gives a mixture 84% pure, and 3 liters of the first mixed with 2 liters of the second gives a mixture 86% pure. Find the per cent of purity of each acid.

40. When weighed in water the crown of Hiero of Syracuse, which was part gold and part silver, and which weighed 20 pounds in air, lost $1\frac{1}{4}$ pounds. How much gold and how much silver did it contain?

HINT. When weighed in water $19\frac{1}{4}$ pounds of gold and $10\frac{1}{2}$ pounds of silver each lose 1 pound.

41. Find the positive integers which satisfy the equation $5x + 2y = 42$.

Solution. $$x = \frac{42 - 2y}{5}.$$

If x is to be a positive integer, $\frac{42-2y}{5}$ must be integral; that is, $42 - 2y$ must be a positive integral multiple of 5. Hence y can only have the values 1, 6, 11, and 16. The corresponding values of x are 8, 6, 4, and 2.

The various related sets of integral values of the unknowns which satisfy an equation may be effectively represented to the eye by the graph of the equation. Since the equation $5x + 2y = 42$ has the integral sets of roots (8, 1), (6, 6), etc., the line of which this is the equation passes through the points whose coördinates are these

sets of integers. If the line does not enter the first quadrant, we can see at a glance that the corresponding equation has no positive integral sets of roots.

42. Solve in positive integers $7x + 2y = 36$, and illustrate the result graphically.

43. In how many ways can a debt of $73 be paid with 5-dollar and 2-dollar bills? Illustrate the result graphically.

44. A man buys calves at $6 each and pigs at $4 each, spending $72. How many of each did he buy?

45. In how many ways can $1.75 be paid in quarters and nickels?

46. A farmer sells some calves at $6 each, pigs at $3 each, and lambs at $4 each, receiving for all $126. In how many ways could he have sold 32 animals at these prices for the same sum? Determine the number of animals in the various groups.

HINT. Form two equations in three unknowns. Then eliminate one of the unknowns.

47. In how many ways can a sum of $2.40 be made up with nickels, dimes, and quarters, on the condition that the number of nickels used shall equal the number of quarters and dimes together? Determine the various groups.

CHAPTER VI

EXPONENTS

51. Fundamental laws of exponents. The four laws of exponents used in the preceding chapters are:

I. Law of Multiplication,

$$x^a \cdot x^b = x^{a+b}.$$

If a and b are positive integers, we have

$$x^a = x \cdot x \cdot x \cdot x \cdots \text{ to } a \text{ factors,}$$

and $$x^b = x \cdot x \cdot x \cdots \text{ to } b \text{ factors.}$$

Hence $$x^a \cdot x^b = (x \cdot x \cdot x \cdots \text{ to } a \text{ factors}) \times (x \cdot x \cdot x \cdots \text{ to } b \text{ factors})$$
$$= x \cdot x \cdot x \cdots \text{ to } a + b \text{ factors}$$
$$= x^{a+b} \text{ by the definition of an exponent.}$$

II. Law of Division,

$$x^a \div x^b = x^{a-b}.$$

Again, if a and b are positive integers, we have

$$x^a \div x^b = \frac{x^a}{x^b} = \frac{x \cdot x \cdot x \cdots \text{ to } a \text{ factors}}{x \cdot x \cdot x \cdots \text{ to } b \text{ factors}}.$$

If b is less than a, the b factors of the denominator may be canceled with b factors of the numerator, leaving $a - b$ factors in the numerator.

Hence $$\frac{x^a}{x^b} = x^{a-b}.$$

If b is greater than a,

$$\frac{x^a}{x^b}=\frac{1}{x^{b-a}}.$$

III. Law of Involution, or raising to a power,

$$(x^a)^b=x^{ab}.$$

As before, if a and b are positive integers, we have

$$\begin{aligned}(x^a)^b &= x^a \cdot x^a \cdot x^a \cdots \text{to } b \text{ factors}\\ &= x^{a+a+a+\cdots} \cdots \text{to } b \text{ terms of the exponent}\\ &= x^{ab}.\end{aligned}$$

IV. Law of Evolution, or extraction of roots,

$$\sqrt[b]{x^a}=x^{\frac{a}{b}}.$$

Law I may be stated more completely thus:

$$x^a \cdot x^b \cdot x^c \cdots = x^{a+b+c+\cdots}.$$

Law III includes the more general forms:

(a) $$(x^a y^b)^c = x^{ac} y^{bc}.$$

(b) $$((x^a)^b)^c \cdots = x^{abc\cdots}.$$

When Laws I, II, and III were used in previous work in multiplication and division, we always assumed that a and b were positive integers and, in Law II, that a was greater than b. In the work on radicals (see "First Course in Algebra," Revised Edition, pp. 251–252) the meaning of an exponent was extended so as to include fractional exponents, as defined by Law IV. Though Laws I–IV have thus far been restricted to positive integers and fractions, they hold, nevertheless, for any rational values of a and b. This fact will be assumed without proof. We shall now explain the meaning which, according to these laws, must be given to a zero or to a negative exponent.

52. Meaning of zero as an exponent. From Law II,

$$x^4 \div x^4 = x^{4-4} = x^0.$$

But $$x^4 \div x^4 = \frac{x^4}{x^4} = 1.$$

Therefore $$x^0 = 1.$$

More generally, $$x^a \div x^a = x^{a-a} = x^0,$$

and, as before, $$x^0 = 1.$$

That is, any number (except zero) whose exponent is zero is equal to 1. Hence $4^0 = (\frac{2}{3})^0 = (-6)^0$, for each equals 1. Again, if x is not zero, $(5x)^0 = 1$, and if x is not 1, $(x^2 - 2x + 1)^0 = 1$.

53. Meaning of a negative exponent. From Law II,

$$a^3 \div a^5 = a^{3-5} = a^{-2}.$$

Obviously, $$a^3 \div a^5 = \frac{a^3}{a^5} = \frac{1}{a^2}.$$

Therefore a^{-2} is another way of writing $\frac{1}{a^2}$.

Then $$4^{-3} = \frac{1}{4^3} = \frac{1}{64}.$$

Also $$a^{-\frac{2}{3}} = \frac{1}{a^{\frac{2}{3}}} = \frac{1}{\sqrt[3]{a^2}}.$$

In general terms, $$x^{-a} = \frac{1}{x^a}.$$

Consequently $$\frac{1}{x^{-a}} = \frac{1}{\frac{1}{x^a}} = x^a.$$

Similarly, we obtain the more general results

$$bx^{-a} = \frac{b}{x^a} \quad \text{and} \quad \frac{b}{x^{-a}} = bx^a.$$

Therefore, *Any factor of the numerator of a fraction may be omitted from the numerator and written as a factor of the denominator, and vice versâ, if the sign of the exponent of the factor be changed.*

It follows that any expression involving negative exponents may be written as an expression involving only positive exponents. That is to say, negative exponents are not a mathematical necessity but merely a convenience. The extension of the laws of exponents which brings with it the zero exponent and the negative exponent is an illustration of what is called the Law of Permanence of Form.

It is to be understood that the part of the rule for multiplication (p. 5) and of the rule for division (p. 7), which determines the exponents in the product or in the quotient, applies to all numbers, whether positive or negative, integral, fractional, or literal. Hence those rules need not be restated here.

ORAL EXERCISES

Fractional, Negative, and Zero Exponents

Simplify:

1. $x^{\frac{1}{2}} \cdot x^{\frac{1}{3}}$.
2. $x^{\frac{2}{5}} \cdot x^{\frac{1}{5}}$.
3. $x^{\frac{1}{2}} \cdot x^{\frac{1}{4}}$.
4. $x^{\frac{1}{3}} \cdot x^{\frac{1}{4}}$.
5. $x^2 \div x^{\frac{1}{2}}$.
6. $x^{\frac{2}{3}} \cdot x^3$.
7. $x^{\frac{1}{4}} \cdot x$.
8. $x^2 \cdot x^0$.
9. $x^a \cdot x$.
10. $x^3 \div x^{-2}$.
11. $x^{\frac{1}{2}} \cdot x^{-\frac{1}{3}}$.
12. $x^{\frac{1}{6}} \cdot x^{-\frac{1}{2}}$.
13. $x^2 \div x^{-2}$.
14. $x^a \div x^{-2}$.
15. $(5\,x)^0 \cdot 5\,x^0$.
16. $7\,x^0 \cdot x^3$.
17. $x^{\frac{1}{2}} \cdot x^a$.
18. $a^{\frac{1}{2}} \cdot a^{\frac{1}{5}} \cdot a^{\frac{1}{2}}$.
19. $\left(a^{\frac{1}{3}} - a^{\frac{1}{5}}\right) \cdot a^{\frac{1}{2}}$.
20. $\left(a^{\frac{1}{2}} + a^{\frac{1}{4}} + 1\right)a^{\frac{1}{4}}$.
21. $x^{3a-1} \cdot x^{2-2a}$.
22. $x^a \div x^2$.
23. $x^{2a-1} \cdot x^{\frac{1}{2}}$.
24. $\sqrt{x} \cdot x$.
25. $\sqrt[3]{x} \div x^2$.
26. $\sqrt{x} \cdot \sqrt[3]{x}$.
27. $\sqrt{x^3} \div \sqrt{x}$.

28. $\sqrt{x} \cdot \sqrt[3]{x^2}$.

29. $\sqrt{x^0} \cdot \sqrt{x^3}$.

30. $x^{a-1} \cdot x^{1-a}$.

31. $x^{2a-3} \div x^{3-2a}$.

32. $(x^2)^3 \cdot x^2$.

33. $(x^{\frac{1}{2}})^3 \cdot x^{\frac{1}{3}}$.

34. $(x^2)^{3a} \cdot x^{-2a}$.

35. $x^3 \div \dfrac{1}{x^{-2}}$.

36. $x^{5a} \div \dfrac{1}{x^{2a}}$.

37. $x^2 \div \dfrac{1}{x^{-2}}$.

38. $x^{-4} \div \dfrac{1}{x^{-6}}$.

39. $x^{1-a} \div x^{-2a}$.

40. $ax^{-3} \div a^{-2}x^2$.

Read the following with positive exponents and simplify the results:

41. m^{-3}.

42. $2\,a^{-3}$.

43. $3\,ab^{-2}$.

44. $7\,x^2y^{-2}$.

45. $x^{-1}y^{-2}z$.

46. $4\,a^3b^{-2}c^2$.

47. $\dfrac{3}{a^{-2}}$.

48. $\dfrac{4\,x}{y^{-3}}$.

49. $\dfrac{4\,c^0}{xy^{-2}}$.

50. $\dfrac{4\,a^{-2}b^0}{y^{-5}}$.

51. $\dfrac{5^{-1}(ab)^0}{10^{-2}b^2}$.

52. $\dfrac{3\,a^3b^{-2}c}{4\,a^{-2}b^0}$.

53. $\dfrac{12\,x^2y^{-1}}{2\,yx^{-1}}$.

54. $\dfrac{10^{-1}a}{bc^3}$.

55. $\dfrac{4\,a^{-2}bc^{-3}}{6\,a^{-2}b^{-3}c^0}$.

56. $\dfrac{4^{-3}r^{-6}s^6}{s^{-2}r^{-2}t^3}$.

57. $\dfrac{2\,s^{-1}}{m^na^{-b}}$.

58. $2\,e^x \div \dfrac{3}{e^{-x}}$.

Arrange terms so that the exponents of one letter occur in the descending order:

59. $a^2 + a^{-1} - 6\,a + 3\,a^0$.

60. $a^2 + a^{-\frac{1}{2}} - a^{\frac{1}{2}} + a + 3\,a^0$.

61. $a + a^3 + 2 + a^{-2} + a^{-1}$.

62. $a^{\frac{1}{2}} + a^{-\frac{1}{3}} + a^{-\frac{2}{3}} + a + 3$.

63. $a^3 + \dfrac{1}{a^2} + a + \dfrac{1}{a} + a^3$.

64. $a^2 + \dfrac{1}{a^3} - 2 + \dfrac{1}{a} + a^3$.

65. $a^3 + \dfrac{1}{b^3} + \dfrac{a}{b^2} + \dfrac{b^{-4}}{a^{-4}}$.

66. $a^2 - \dfrac{c^3}{a^2} + \dfrac{a}{c^2} - \dfrac{3\,c}{a} + 5$.

67. Arrange the polynomials in Exercises 17 and 21 on page 97 in descending order with respect to the letter a, and Exercises 22 (p. 97) and 28 (p. 98) with respect to x.

EXERCISES IN MULTIPLIC

Perform the indicated multiplication

1. $(x + x^{\frac{1}{2}} + x^{-\frac{1}{2}})2x^{\frac{3}{2}}$.

2. $(a^{\frac{1}{2}}x + ax^{\frac{1}{2}} - a^{\frac{1}{2}}x^{-\frac{1}{2}})ax^{\frac{1}{2}}$.

3. $(x^4 - a^4)x^{-2}a^{-2}$.

4. $(x^2 - 5ax + 6a^2)a^{\frac{2}{3}}x^{-\frac{2}{3}}$.

5. $(x^{\frac{1}{3}} + y^{\frac{1}{3}})x^{\frac{2}{3}}y^{\frac{1}{3}}$.

6. $(a^{\frac{1}{2}} - x^{\frac{1}{2}})(a^{\frac{1}{2}} + x^{\frac{1}{2}})$.

7. $(x^{\frac{1}{3}} + y^{\frac{1}{3}})(x^{\frac{1}{3}} - y^{\frac{1}{3}})$.

8. $(a^{-2} + 3)(a^{-2} - 5)$.

9. $(x^{2a} - 3x^{a})(x^2 - 2x)$.

10. $(a^{-1} - a)^2$.

11. $(a^3 - 2a^{-2})^3$.

12. $(a^{-1} - 2a + 3a^{-2})^2$.

13. $(e$

14. $(e$

15. $($

16. $($

17.

18.

19.

20.

21

2

2

25. $(x^{\frac{1}{3}} + 2y^{\frac{1}{3}})(x^{\frac{2}{3}} -$

26. $(a + a^{\frac{1}{2}}b^{\frac{1}{2}} + b)(a$

27. $(a^{\frac{1}{2}} - a^{\frac{1}{4}}b^{\frac{1}{4}} + b^{\frac{1}{2}})($

28. $(x + x^{\frac{1}{2}}y^{-\frac{1}{2}} + y^{-}$

29. $(\sqrt[3]{a^5} - 5\sqrt{a})($

HINT. This is best wr

30. $(5\sqrt{c^{-5}} + \sqrt[3]{d^{-}}$

31. $(x^{\frac{2}{3}} - a^{\frac{1}{2}})^2(\sqrt[3]{x}$

32. $\left(\sqrt{a} + \frac{\sqrt{b}}{c}\right)($

EXERCISES IN DIVISION

Perform the indicated division and simplify:

1. $x^4 \div x^6$.

2. $x^3 \div x^{\frac{1}{3}}$.

3. $x^{\frac{1}{2}} \div x^2$.

4. $ax^{\frac{2}{3}} \div a^{\frac{1}{2}}x^{\frac{1}{2}}$.

5. $\dfrac{ax - a^2x^3}{a^2x^{\frac{1}{2}}}$.

6. $(x^a - 2x^{2a-1} + 3x^{3a-2}) \div x^{2a-1}$.

7. $(6a^{3+4n} - 9a^{n-2} + 12a^{2-n}) \div 3a^{n-2}$.

8. $(x - y) \div \left(x^{\frac{1}{2}} - y^{\frac{1}{2}}\right)$.

9. $(x + y) \div \left(x^{\frac{1}{3}} + y^{\frac{1}{3}}\right)$.

10. $(x - 8y) \div \left(x^{\frac{1}{3}} - 2y^{\frac{1}{3}}\right)$.

11. $(16x^2 - 4096y^2) \div \left(2x^{\frac{1}{2}} + 8y^{\frac{1}{2}}\right)$.

12. $\left(a^{\frac{3}{2}} + b\right) \div \left(\sqrt{a} + \sqrt[3]{b}\right)$.

13. $(a^2 + ab^{-1} + b^{-2}) \div \left(a - a^{\frac{1}{2}}b^{-\frac{1}{2}} + b^{-1}\right)$.

14. $(e^{2x} + e^{-2x} + 2) \div (e^{-x} + e^x)$.

15. $\left(3e^x + \dfrac{1}{e^{3x}} + e^{3x} + \dfrac{3}{e^x}\right) \div (e^x + e^{-x})$.

16. $(6 + e^{-4x} + e^{4x} - 4e^{-2x} - 4e^{2x}) \div (e^x - e^{-x})$.

17. $\left(a + 2b + 2a^{\frac{2}{5}}b^{\frac{1}{3}} + a^{\frac{3}{5}}b^{\frac{2}{3}}\right) \div \left(a^{\frac{3}{5}} + 2b^{\frac{1}{3}}\right)$.

18. $(m^4 - 7m^{-2} + m^{-4} + 7m^2 + 8) \div (5 - m^{-2} + m^2)$.

19. $(9x^{5n-4} - x^{3n-2} + 2x^{2n-1}) \div (2x^{n-1} + 3x^{2n-2})$.

20. $\left(16x - 8y^{\frac{1}{2}} + x^{-\frac{5}{2}}y - 2x^{-\frac{3}{2}}y^{\frac{1}{2}}\right) \div \left(x^{-\frac{1}{2}}y^{-\frac{1}{2}} - 8x^2y^{-1}\right)$.

21. $\left(40ab - 25a^{-\frac{9}{5}}b^{\frac{5}{4}} - 16a^{\frac{19}{5}}b^{\frac{3}{4}}\right) \div \left(-4a^{\frac{12}{5}}b^{-\frac{5}{8}} + 5a^{-\frac{2}{5}}b^{-\frac{3}{8}}\right)$

22. $(2x^{-2a} - 28x^{-a} + 33 + 11x^{2a} + 38x^a + x^{3a})$
$\div (4 + x^a - 2x^{-a})$.

Divide:

23. $a^{\frac{3}{2}} - ab^{\frac{1}{2}} + a^{\frac{1}{2}}b - b^{\frac{3}{2}}$ by $a^{\frac{1}{2}} - b^{\frac{1}{2}}$.

24. $3x^{-10} + x^6 - 4x^{-6}$ by $2x^{-2} + x^2 + 3x^{-6}$

25. $x^{\frac{3}{5}} - y^{\frac{3}{5}}$ by $\left[\left(x^{\frac{1}{5}} - y^{\frac{1}{5}}\right) \div \left(x^{\frac{1}{10}} + y^{\frac{1}{10}}\right)\right]$.

26. $9m + 4m^{-1} - 13$ by $3m^{\frac{1}{2}} - 5 + 2m^{-\frac{1}{2}}$.

27. $x^{2a} + 4x^{-2a} - 29$ by $x^{a} - 2x^{-a} - 5$.

28. $9x^{2a} + 25x^{-4a} - 19x^{-a}$ by $5x^{-2a} + 3x^{a} - 7x^{-\frac{a}{2}}$.

29. $\left(\dfrac{x^{\frac{3}{2}}}{8} + \dfrac{27y^{\frac{3}{2}}}{64}\right) \div \left[\left(\dfrac{3x^{\frac{1}{2}}}{2} + \dfrac{9y^{\frac{1}{2}}}{4}\right)\left(64x - 96x^{\frac{1}{2}}y^{\frac{1}{2}} + 144y\right)\right]$.

NOTE. To us, who use the notation of exponents every day, it seems so simple and natural a method of expressing the product of several factors that it is difficult to understand why such a long time was necessary to develop it. But here, as in many other instances, it required a great man to discover what to us seems the most obvious relation. The man who brought the notation of exponents to its modern form was John Wallis (1616–1703), an Englishman.

Though the idea of using negative and fractional exponents had occurred to writers before Wallis, it was he who showed their naturalness, and who introduced them permanently. He also was the first to use the ordinary sign ∞ to denote infinity.

MISCELLANEOUS EXERCISES ON EXPONENTS

Find the numerical value of the following:

1. 3^{-2}.
2. 4^{-3}.
3. $2^{-4} \cdot 3^{0}$.
4. $2^{-2} \cdot 3^{-4}$.
5. $7 \cdot 7^{0} \cdot 0$.
6. $(\frac{1}{2})^{-3}$.
7. $(\frac{2}{3})^{-2} \cdot 4^{0}$.
8. $(\frac{4}{5})^{-3} \cdot (\frac{10}{3})^{-2}$.
9. $\dfrac{2}{3^{-2}}$.
10. $\dfrac{3}{3^{0}}$.
11. $\dfrac{12}{4^{-1}}$.
12. $5 \cdot 2^{0} - (5 \cdot 2)^{0}$.
13. $\dfrac{4^{-2} \cdot 3^{-2}}{6^{-2}}$.
14. $32^{-\frac{2}{5}}$.
15. $0^{5} \cdot 5^{0}$.
16. $4^{-\frac{1}{2}}$.
17. $8^{-\frac{1}{3}}$.
18. $16^{-\frac{1}{4}}$.
19. $8^{-\frac{2}{3}}$.
20. $16^{-\frac{3}{2}}$.
21. $25^{1.5}$.
22. $(-64)^{-\frac{2}{3}}$.
23. $(-32)^{\frac{1}{5}}$.
24. $(32)^{-4}$.
25. $(-125)^{-\frac{2}{3}}$.
26. $\sqrt[3]{27^{-2}}$.
27. $\sqrt[3]{8^{-2}}$.

28. $(\sqrt[3]{-8})^2$.

29. $(\frac{1}{2})^{-4} \cdot (\frac{1}{3})^{-3} \cdot (\frac{1}{2})^0$.

30. $(\frac{1}{6})^{-2}$.

31. $(.04)^{\frac{3}{2}}$.

32. $(.027)^{-\frac{2}{3}}$.

33. $(.064)^{-\frac{1}{3}}$.

34. $(.00032)^{\frac{2}{5}}$.

35. $\dfrac{\sqrt[3]{9^{-3}} \cdot \sqrt[2]{9^{-2}}}{3^{-4}}$.

36. $\dfrac{2^{-1}}{2^{-2} - 2^{-3}}$.

HINT. $\dfrac{2^{-1}}{2^{-2} - 2^{-3}} = \dfrac{\frac{1}{2}}{\frac{1}{2^2} - \frac{1}{2^3}}$, etc.

37. $\dfrac{3^{-2} - 2^{-2}}{3^{-1} - 2^{-1}}$.

38. $\dfrac{2^{-1} + 3^{-1}}{2^{-3} + 3^{-3}}$.

39. $\dfrac{3^{-3} - 2^{-3}}{3^{-1} - 2^{-1}}$.

Write with positive exponents and simplify the results:

40. $\dfrac{2}{a^{-2} - b^{-2}}$.

HINT. $\dfrac{2}{a^{-2} - b^{-2}} = \dfrac{2}{\frac{1}{a^2} - \frac{1}{b^2}}$, etc.

41. $\dfrac{3}{a^{-1} + b^{-1}}$.

42. $\dfrac{a}{a^{-2} - b^{-2}}$.

43. $\dfrac{5se^2}{s^{-2} + e^{-2}}$.

44. $\dfrac{a^{-2}}{a^{-2} + b^{-2}}$.

45. $\dfrac{a^{-3}b^{-3}}{a^{-3} + b^{-3}}$.

46. $\dfrac{a^{-2} - b^{-2}}{a^{-1} + b^{-1}}$.

47. $\dfrac{a^{-4} - b^{-4}}{a^{-2} - b^{-2}}$.

48. $\dfrac{a^{-1} + b^{-1}}{a^{-3} + b^{-3}}$.

49. $\dfrac{a^{-3} - 27^{-1}}{a^{-1} - 3^{-1}}$.

Write without a denominator and simplify the results:

50. $\dfrac{2xy}{z^2}$.

51. $\dfrac{4a^3}{b^3}$.

52. $\dfrac{3x}{a^{-2}b^4}$.

53. $\dfrac{4sc^{-3}}{2^{-1}s^{-2}}$.

54. $\dfrac{12a^2b^3}{4xy^2}$.

55. $\dfrac{7x^{-1}y^2}{2^{-1}y^3}$.

56. $\dfrac{5ac^{-2}}{c(x-y)^2}$.

57. $\dfrac{7m^{-3}n^{\frac{1}{2}}}{m^{\frac{2}{3}}(m-n)^0}$.

58. $\dfrac{1}{5a^2(c+d)^{-3}}$.

59. $\dfrac{a(x-y)^{-2}}{bcx(x-y)}$.

60. $\dfrac{42m^{-n}n^{2m}}{56m^{-2n}n^{-3m}}$.

61. $\dfrac{r^{-1}s^2}{r^{-2}s^3(s-r)^3}$.

Simplify:

62. $(2^3)^3$.

63. $(2^3)^{-2}$.

64. $(2^{-3})^{-2}$.

65. $\left[\left(\frac{3}{2}\right)^{-3}\right]^2$.

66. $(x^2)^3$.

67. $\left(x^{\frac{1}{2}}\right)^4$.

68. $\left(x^{\frac{1}{3}}\right)^{\frac{1}{2}}$.

69. $(x^2)^{-2}$.

70. $(x^{-3})^{-2}$.

71. $\left(x^{-\frac{1}{2}}\right)^{-\frac{1}{3}}$.

72. $(3x^{-2})^3$.

73. $(5a^2)^{-3}$.

74. $(c^{-2}d)^2$.

75. $(5^0 \cdot 2^6 \cdot 3^3)^{\frac{1}{3}}$.

76. $(25a^4b^6)^{-\frac{1}{2}}$.

77. $(2x^5)^0 \cdot 8 \cdot 4^{\frac{1}{2}}$.

78. $(a^2b)^3(a+b)$.

79. $(a^2)^{3x} \cdot a^{3x}$.

80. $(a^3)^{2x} \cdot (a^2)^{3x}$.

81. $(a^{x+1})^2 \cdot (a^{1-x})^2$.

82. $(a^3)^{x+y} \cdot (a^2)^{y-x}$.

83. $(a^2b)^x \cdot (b^2a^3)^{2x}$.

84. $(x^2 - x^{-2})x^3$.

85. $(x^2 + xy^{-2})x^{-2}$.

86. $2^3 \cdot 4^2 = 2^?$.

87. $2^n \cdot 4^n = 2^?$.

88. $\dfrac{2^n \cdot 8^{2n}}{4^{3n}} = 2^?$.

89. $2^n \cdot 4^{n+1} \div 2^n = 2^?$.

90. $\dfrac{4^{n+1}}{2^n(4^{n-1})^n} \div \dfrac{8^{n+1}}{(4^{n+1})^{n-1}} + 5 = ?$

Solve for n:

91. $8^n \cdot 4^{2n} = 2^{14}$.

92. $3^6 \cdot 9^n = 81^2$.

93. $9^n \cdot 3^3 = 27^n$.

94. $2^{2n+2} \cdot 4^{n+2} = 8^{2n}$.

95. $81 \cdot 27^n = (9^n)^{\frac{n}{2}}$.

96. $(25^n)^n = \dfrac{5^{7n}}{(125)^2}$.

97. $2^{6n+3} \cdot 4^{3n+6} = (8^n)^n$.

Solve for x:

98. $x^{\frac{3}{2}} = 8$.

99. $x^{-\frac{1}{3}} = 5$.

100. $x^{-\frac{4}{3}} = 256$.

101. $x^{-\frac{1}{2}} = 2$.

102. $x^{\frac{2}{3}} = 4$.

103. $x^{-\frac{2}{3}} = 4$.

104. $x^{\frac{1}{4}} = 2$.

105. $x^{\frac{4}{3}} = 16$.

106. $x^{-\frac{2}{3}} = 32$.

107. $x^{\frac{3}{2}} = 343$.

108. $\frac{1}{2}x^{-\frac{2}{3}} = 2$.

109. $\left(x^{-\frac{1}{2}}\right)^{-4} = 49$.

110. $\left(ax^{\frac{1}{2}}\right)^{-6} = 27$.

111. $\dfrac{\sqrt[3]{x^{\frac{7}{5}}}}{\sqrt[3]{x^{\frac{1}{5}}}} = \dfrac{\sqrt[5]{25}}{\sqrt[5]{16}}$.

CHAPTER VII

SQUARE ROOT

54. Square root. The **square root** of any number is one of the two equal factors whose product is the number.

From the law of signs in multiplication it follows that

Every positive number or algebraic expression has two square roots which have the same absolute value but opposite signs.

55. Square root of a monomial. For extracting the square roots of any monomial we have the

Rule. *Write the square root of the numerical coefficient preceded by the sign* $\pm$ *and followed by all the letters of the monomial, giving to each letter an exponent equal to one half its exponent in the monomial.*

A rule similar to the preceding one holds for the fourth root, the sixth root, and other even roots.

56. Cube root. The cube root of any number is one of the three equal factors whose product is the number.

For extracting the cube root of a monomial we have the

Rule. *Write the cube root of the numerical coefficient followed by all the letters of the monomial, giving to each letter an exponent equal to one third of its exponent in the monomial.*

A rule similar to the preceding one holds for the fifth root, the seventh root, and other odd roots.

57. Principal root. For a given index the **principal root** of a number is its one real root if it has but one, or its positive real root if it has two real roots of that index.

Then the principal square root of 9 is $+3$; the principal fourth root of 16 is $+2$, not -2. The square root of -4 or -9 is *not* real; such numbers have no principal square root.

The principal cube root of 8 is 2, of -27 is -3. The principal fifth root of 32 is $+2$, of -32 is -2.

Every number has more than one root of given odd index. The number 8, for example, has two other cube roots besides its principal cube root 2. What they are and how they are obtained will be made clear in the chapter on Imaginaries, where the consideration of the square roots of negative numbers will also be taken up.

Unless otherwise specified, only the principal odd root of a number will be considered.

ORAL EXERCISES

Find the principal square root of the following:

1. 16. **2.** 25. **3.** $4a^2$. **4.** $9a^4$. **5.** $36a^6$. **6.** $49t^8$. **7.** $64t^4$. **8.** $81x^6$. **9.** $\frac{1}{4}$. **10.** $\frac{1}{a^2}$. **11.** $\frac{9}{a^4}$. **12.** $\frac{x^2}{a^2}$. **13.** x^{-4}. **14.** a^2x^{-2}. **15.** $9x^4x^{-6}$. **16.** $4x^0$. **17.** $x^{\frac{4}{3}}$. **18.** $x^{\frac{2}{5}}$.

Find the principal cube root of the following:

19. 8. **20.** 27. **21.** 64. **22.** $8a^3$. **23.** $8a^6$. **24.** $27x^9$. **25.** $64x^6$. **26.** -8. **27.** -27. **28.** -64. **29.** $-125a^3$. **30.** $-27a^6$. **31.** $-64a^3$. **32.** $-8a^9$. **33.** $216a^3$. **34.** $343a^6$. **35.** $-512a^3$. **36.** $-27a^{\frac{4}{3}}$. **37.** $-8x^{-3}$. **38.** $-27x^{-6}$.

Find the principal fourth root of the following:

39. 16. **40.** 81. **41.** 256. **42.** a^4. **43.** x^8. **44.** x^{12}. **45.** $16x^4$. **46.** $625x^8$. **47.** $16x^{-4}$. **48.** $x^{\frac{4}{3}}$. **49.** $16x^{-\frac{4}{3}}$. **50.** $81x^{-\frac{4}{5}}$.

Give the principal root and one other root for the following:

51. The fourth root of 81; of a^4; of x^{-4}.

52. The sixth root of 64; of a^6; of x^{-6}.

53. What is the sign of the principal *odd* root of a positive number? The principal odd root of a negative number?

54. What is the sign of the principal even root of a positive number?

55. State the rule for extracting the fourth root of a monomial.

56. State the rule for extracting the fifth root of a monomial.

57. Can one obtain the fifth root of a monomial by extracting the square root of its cube root? by extracting the cube root of its square root? Explain.

58. Square root of polynomials. Extracting the square root of a number is essentially an undoing of the work of multiplication. The square of any polynomial may be represented by

$$\begin{aligned}(h+t+u)^2 &= h^2+2\,ht+t^2+2\,hu+2\,tu+u^2\\ &= h^2+(2\,h+t)\,t+(2\,h+2\,t+u)\,u.\end{aligned}$$

A little study of this last form and a comparison with the example which follows will make clear the reason for each step of the process.

EXAMPLE

	$h^2+2\,ht+t^2+2\,hu+2\,tu+u^2$	$h+t+u$
	h^2	
First trial divisor, $2\,h$	$+2\,ht+t^2$	
First complete divisor, $2\,h+t$	$+2\,ht+t^2$	$=(2\,h+t)\,t$
Second trial divisor, $2\,h+2\,t$	$+2\,hu+2\,tu+u^2=(2\,h+2\,t+u)\,u$	
Second complete divisor, $2\,h+2\,t+u$	$+2\,hu+2\,tu+u^2=(2\,h+2\,t+u)\,u$	

Therefore the required square roots are $\pm\,(h+t+u)$.

EXERCISES

Extract the square root of the following:

1. $x^4 + 4x^3 - 2x^2 - 12x + 9.$
2. $a^6 - 10a^4 - 4a^3 + 25a^2 + 20a + 4.$
3. $x^6 + 4x^2 + 16 - 4x^4 + 8x^3 - 16x.$
4. $t^6 + 4t^4 + 9 + 4t^5 - 6t^3 - 12t^2.$
5. $4a^4 + 9a^2t^2 + t^4 + 12a^3t - 4a^2t^2 - 6at^3.$
6. $4a^8 + 12a^4 - 7 - 24a^{-4} + 16a^{-8}.$
7. $49c^{-6} - 28c^{-4} + 74c^{-2} - 20 + 25c^2.$
8. $9x^2 + 4x + 1 + 12x^{\frac{3}{2}} + 6x + 4x^{\frac{1}{2}}.$
9. $9x^4 - 6x^{\frac{7}{2}} + x^3 - 66x^{\frac{5}{2}} + 22x^2 + 121x.$
10. $25x^3 - 10x^2 + 90x^{\frac{7}{4}} + x - 18x^{\frac{3}{4}} + 81x^{\frac{1}{2}}.$
11. $16m^{-7} - 8m^{-4} + 104m - 26m^4 + 169m^9 + m^{-1}.$
12. $\dfrac{4x^2}{y^2} + 7 + \dfrac{y^2}{4x^2} + \dfrac{12x}{y} - \dfrac{3y}{x}.$

Solution. Arranging terms in descending powers of x and applying the method of page 103, we obtain the following:

$$
\begin{array}{lll}
 & \dfrac{4x^2}{y^2} + \dfrac{12x}{y} + 7 - \dfrac{3y}{x} + \dfrac{y^2}{4x^2} & \Big| \ \dfrac{2x}{y} + 3 - \dfrac{y}{2x} \\
\left(\dfrac{2x}{y}\right)^2 = & \dfrac{4x^2}{y^2} & \\
2\left(\dfrac{2x}{y}\right) = \dfrac{4x}{y} & \quad \dfrac{12x}{y} + 7 & \\
\quad \dfrac{4x}{y} + 3 & \quad \dfrac{12x}{y} + 9 & = \left(\dfrac{4x}{y} + 3\right)3 \\
2\left(\dfrac{2x}{y} + 3\right) = \dfrac{4x}{y} + 6 & \quad -2 - \dfrac{3y}{x} + \dfrac{y^2}{4x^2} & \\
\quad \dfrac{4x}{y} + 6 - \dfrac{y}{2x} & \quad -2 - \dfrac{3y}{x} + \dfrac{y^2}{4x^2} & = \left(\dfrac{4x}{y} + 6 - \dfrac{y}{2x}\right)\left(-\dfrac{y}{2x}\right)
\end{array}
$$

Therefore the square roots are $\pm\left(\dfrac{2x}{y} + 3 - \dfrac{y}{2x}\right).$

13. $x^4 + 6x^3 + \dfrac{29x^2}{3} + 2x + \dfrac{1}{9}$.

14. $4a^4 + \dfrac{4a^3}{3} - \dfrac{35a^2}{9} - \dfrac{2a}{3} + 1$.

15. $\dfrac{25x^4}{4} - \dfrac{127x^2}{18} - 2x + \dfrac{10x^3}{3} + 2\frac{1}{4}$.

16. $\dfrac{a^2}{4} + \dfrac{b^2}{9} + 36 + \dfrac{ab}{3} - 6a - 4b$.

17. $\frac{9}{4}a^4 - 2a^3 + 7\frac{4}{9}a^2 - \frac{28}{9}a + \frac{49}{9}$.

18. $9c^4 - 12c^3 + 4c^2 - \dfrac{4}{c} + \dfrac{1}{c^4} + 6$.

19. $\dfrac{a^2}{9} - 18 + \dfrac{9}{a^2} + 9a^2 - 2 + 2a^2$.

20. $\dfrac{x^2}{a^2} - \dfrac{5a}{x} + \dfrac{5x}{a} + \dfrac{a^2}{x^2} + 4\frac{1}{4}$.

21. $\dfrac{a^2}{b^2} + \dfrac{b^2}{a^2} + a^2b^2 + 2a^2 + 2 + 2b^2$.

22. $\dfrac{a^2}{x^2} + \dfrac{x^2}{a^2} - \dfrac{2a}{x} - \dfrac{2x}{a} + 3$.

23. $\dfrac{a^4}{25c^4} + \dfrac{2a^2}{c^3} + \dfrac{117}{5c^2} - \dfrac{40}{a^2c} + \dfrac{16}{a^4}$.

24. $\dfrac{a^2}{4c^4} + 9 + \dfrac{4c^2}{25a^4} - \dfrac{3a}{c^2} + \dfrac{2}{5ac} - \dfrac{12c}{5a^2}$.

25. $\dfrac{a^2}{9c^2} + \dfrac{16c^2}{a^2} + \dfrac{4a}{3c} + \dfrac{16c}{a} + \dfrac{20}{3}$.

26. $\dfrac{1}{4a^4} - \dfrac{3x}{a^3} + \dfrac{9x^2}{a^2} + \dfrac{a^4}{25x^2} - \dfrac{6a}{5} + \dfrac{1}{5x}$.

Find the first four terms in the square root of the following:

27. $1 + 2x$.

28. $\frac{25}{9} + a^3$.

59. Square root of arithmetical numbers. The abbreviated process of extracting the square roots of an arithmetical number is as follows:

```
          7'32'67'89|2706.8+
          4
      47 |332
         |329
    5406   |36789
           |32436
    54128    |435300
             |433024
                 2276
```

Therefore the square roots of 7326789 are $\pm 2706.8+$.

It follows from the preceding example that the work of extracting the positive square root of a number may be a never-ending process. The number 7,326,789 has no exact square root, and no matter how far the work is carried, there is no final digit. As the work stands we know that the required root lies between 2706.8 and 2706.9.

The method just illustrated for extracting the positive square root of a number is the one commonly used. For it we have the

Rule. *Begin at the decimal point and point off as many periods of two digits each as possible: to the left if the number is an integer, to the right if it is a decimal, to both the left and the right if the number is part integral and part decimal.*

Find the greatest integer whose square is equal to or less than the left-hand period and write this integer for the first digit of the root.

Square the first digit of the root, subtract its square from the first period and annex the second period to the remainder.

Double the part of the root already found, for a trial divisor, divide it into the remainder (omitting from the latter the right-hand digit), and write the integral part of the quotient as the next digit of the root.

Annex the root digit just found to the trial divisor to make the complete divisor, multiply the complete divisor by this root digit, subtract the result from the dividend, and annex to the remainder the next period, thus making a new dividend.

Double the part of the root already found, for a new trial divisor, and proceed as before until the desired number of digits of the root have been found.

After extracting the square root of a number involving decimals, point off one decimal place in the root for every decimal period in the number.

Check. *If the root is exact, square it. The result should be the original number. If the root is inexact, square it and add to this result the remainder. The sum should be the original number.*

EXERCISES

Find the positive square root of the following:

1. 6241.	**5.** 53.29.	**9.** 2,932,900.
2. 9216.	**6.** 1.4641.	**10.** 7,049,025.
3. 15,129.	**7.** 216.09.	**11.** 3.9601.
4. 56,169.	**8.** 988,036.	**12.** .0061504.

Find to three decimal places the square root of the following:

13. 7.	**15.** .01235.	**17.** $\frac{5}{7}$.	**19.** $\frac{10}{7}$.	**21.** $23\frac{6}{11}$.
14. .63.	**16.** .96384.	**18.** $4\frac{2}{9}$.	**20.** $\frac{4}{5}$.	**22.** $89\frac{1}{3}$.

23. Find the hypotenuse of a right triangle whose legs are 136 and 273 respectively.

24. A baseball diamond is a square 90 feet on each side. Find the distance from the home plate to second base, correct to .01 of a foot.

25. The hypotenuse of a right triangle is 207 feet, and one leg is 83 feet. Find the other leg, correct to .01 of a foot.

26. The hypotenuse and one leg of a right triangle are respectively 5471 and 4059. Find the other leg.

27. The side of an equilateral triangle is 17 inches. Find its altitude, correct to .1 of an inch.

28. Find the side of an equilateral triangle whose altitude is 15 inches, correct to .001 of an inch.

Fact from Geometry. If a, b, and c represent the sides of a triangle and s equals one half of $a + b + c$, the area of the triangle equals $\sqrt{s(s-a)(s-b)(s-c)}$.

29. Find the area of a triangle whose sides are 12, 27, and 35 inches respectively, correct to .001 of a square foot.

30. By the method of Exercise 29 find to .01 of a square inch the area of a triangle each side of which is 22 inches.

31. Find to two decimals the sum of all of the diagonal lines that can be drawn on the faces of a cube whose edge is 11 inches.

32. Find to two decimals the radius of a circle whose area is 70 square feet.

33. Find to two decimals the diagonal of a room whose dimensions in feet are 15, 22, and 28.

34. Find to two decimals the diagonal of a cube whose edge is 8 feet.

35. A room is 24 feet by 40 feet by 14 feet. What is the length of the shortest broken line from one lower corner to the diagonally opposite upper corner, the line to be on the walls or the floor, but not through the air?

CHAPTER VIII

RADICALS

60. Radical. A **radical** is an indicated root of an algebraic or arithmetical expression.

Thus $\sqrt{9}$, $\sqrt{5}$, $\sqrt[3]{2x}$, and $\sqrt{x^2 - x - 12}$ are radicals.

61. Index. The small figure like the 3 in $\sqrt[3]{7}$ is called the **index** of the radical.

The index determines the order of the radical and indicates the root to be extracted.

For square root the index is usually omitted. Thus $\sqrt{2}$ and $\sqrt{18}$ mean $\sqrt[2]{2}$ and $\sqrt[2]{18}$ respectively.

62. Radicand. The **radicand** is the number or expression under the radical sign. In $\sqrt{5}$ and $\sqrt[3]{2ax}$ the respective radicands are 5 and $2ax$.

63. Fractional exponents. Radical expressions may be written in either of two ways: with radical signs or with fractional exponents.

Thus $\sqrt{5}$ and $5^{\frac{1}{2}}$ have the same meaning, and $\sqrt[3]{a^2}$ equals $a^{\frac{2}{3}}$, etc.

64. Rational numbers. A **rational number** is a positive or a negative integer or any number which can be expressed as the quotient of two such integers.

Thus 7, -6, $\frac{2}{3}$, or 2.871 are rational numbers.

65. Irrational numbers. Any real number which is not rational is **irrational**. (See section 67.)

If a number under a radical sign is such that the root indicated cannot be exactly obtained, the radical represents an irrational number.

For example, $\sqrt{7}$ and $\sqrt[3]{4}$ are irrational. Approximate values for these are given on page 274.

A repeating decimal, though endless, is not an irrational number, for any repeating decimal can be expressed as a common fraction, and is therefore rational.

Thus the repeating decimal $.272727\cdots$ is not irrational, as it exactly equals $\frac{3}{11}$. Similarly, $.2857142857142\cdots$ exactly equals $\frac{2}{7}$, etc.

NOTE. The number $\frac{2}{7}$ reduced to a decimal repeats the digits in groups of six each, and the mere fact that a decimal does so repeat is proof that it is a rational number. On the other hand, the number π is known to be irrational, and its value has been computed to 707 decimals, showing, of course, no repetition. The fact that it does not repeat in 700 digits is, however, no proof that π is irrational, for decimals with even more than that many digits do repeat. For example, the fraction $\frac{10000}{7699}$ equals the decimal $1.29+$, which repeats in groups of 7698 digits each.

66. Imaginary. An indicated *square* root of a *negative* number is called an **imaginary** number.

Thus $\sqrt{-4}$, $\sqrt{-7}$, and $\sqrt{-12}$ are imaginary numbers. And $3+\sqrt{-1}$ is also imaginary, though, as will be seen later (Chapter XII), such numbers are better called complex numbers.

67. Classification of numbers. All the numbers of algebra then may be placed in one or the other of two classes: *real* numbers and *imaginary* numbers.

Real numbers, as we have seen, are of two kinds, *rational* numbers and *irrational* numbers.

68. Surd. A **surd** is an irrational number in which the radicand is rational.

Thus $\sqrt{3}$, $\sqrt[3]{9}$, etc., are surds. But $\sqrt{2+\sqrt{3}}$ and $\sqrt{\pi}$ are not surds.

69. The algebraic sign of a radical. The square root of 4 is both $+2$ and -2. The symbol $\sqrt{4}$, however, signifies only $+2$, the principal root (section 57). Similarly, $\sqrt[4]{81}$ is $+3$ and $\sqrt{64}$ is $+2$. But $-\sqrt{9}$ is -3, and $-\sqrt[4]{16}$ is -2. The symbol $\pm\sqrt{25}$ denotes both $+5$ and -5. Further, $+\sqrt[3]{27}=+3$, $-\sqrt[3]{27}=-3$, and $-\sqrt[3]{-27}=+3$.

The foregoing remarks apply also to fractional exponents. Thus $4^{\frac{1}{2}}=+2$ only, and $81^{\frac{1}{4}}=+3$ only, etc. It should be noted that these statements really define the meaning of such symbols as $\sqrt{\ }$, $\sqrt[4]{\ }$, $\sqrt[6]{\ }$, etc. Such an understanding as this avoids all the ambiguity which would arise if $\sqrt{16}$ meant both $+4$ and -4. The distinctions here made are especially needed in radical equations.

ORAL EXERCISES

Find the numerical value of the following:

1. $\sqrt{4}$.
2. $-\sqrt{9}$.
3. $\sqrt{16}$.
4. $\sqrt[3]{8}$.
5. $\sqrt[3]{-8}$.
6. $-\sqrt[4]{16}$.
7. $-\sqrt[4]{81}$.
8. $\sqrt[5]{32}$.
9. $\sqrt[3]{-125}$.
10. $-\sqrt[5]{-32}$.
11. $\sqrt[6]{64}$.
12. $-\sqrt[4]{625}$.
13. $8^{\frac{1}{3}}$.
14. $36^{\frac{1}{2}}$.
15. $49^{\frac{1}{2}}$.
16. $8^{\frac{2}{3}}$.
17. $16^{\frac{1}{4}}$.
18. $27^{\frac{2}{3}}$.
19. $16^{\frac{3}{4}}$.
20. $(\frac{1}{4})^{\frac{1}{2}}$.
21. $(\frac{4}{9})^{\frac{1}{2}}$.
22. $(-64)^{\frac{1}{3}}$.
23. $(49)^{\frac{3}{2}}$.
24. $(121)^{\frac{3}{2}}$.

Read in radical form:

25. $x^{\frac{3}{2}}$.
26. $x^{\frac{4}{3}}$.
27. $(at)^{\frac{3}{2}}$.
28. $(3t)^{\frac{1}{3}}$.
29. $3t^{\frac{1}{2}}$.
30. $3rx^{\frac{1}{2}}$.
31. $4ax^{\frac{2}{3}}$.
32. $2a^{\frac{1}{2}}x^{\frac{1}{3}}$.
33. $5ax^{\frac{3}{4}}$.
34. $4a^2x^{\frac{3}{2}}$.
35. $a(a-1)^{\frac{1}{2}}$.
36. $2c(2x-3)^{\frac{2}{3}}$.
37. $x^{\frac{1}{n}}$.
38. $2^{\frac{1}{3}}x^{\frac{1}{a}}$.
39. $x^{\frac{a}{n}}t^{\frac{2n}{a}}$.

Read with fractional exponents:

40. $\sqrt{a^5}$.	**43.** $\sqrt[3]{a^2x^4}$.	**46.** $\sqrt[3]{(a+x)^2}$.
41. $\sqrt{ax^3}$.	**44.** $2\sqrt{a}$.	**47.** $\sqrt{(3x-a)^4}$.
42. $\sqrt[3]{a^4}$.	**45.** $\sqrt[3]{2a}$.	**48.** $\sqrt[3]{x^2(y-1)}$.

49. What are the two square roots of 36?

50. What are two fourth roots of 16? of 81? of 625?

51. What is the value of $\sqrt[4]{16}$? of $\sqrt[4]{81}$? of $\sqrt[4]{625}$?

52. What are two sixth roots of 64?

53. What is the distinction between a rational number and an irrational one?

54. Which of the numbers 8, $\frac{2}{3}$, 343, $\sqrt{4}$, $\sqrt{3}$, and π ($\pi = 3.14159+$) are rational? Which are irrational?

55. Give a geometrical illustration of an irrational number by means of a right triangle.

56. Is a radical always a surd? Illustrate.

57. Is a surd always a radical? Illustrate.

58. Distinguish between a surd and a radical.

59. Which of the numbers $\sqrt{3}$, $\sqrt{4}$, $\sqrt[3]{27}$, $\sqrt{\sqrt[3]{6}}$, $\sqrt{2+\sqrt{3}}$, and $\sqrt{2\pi}$ are surds? Which are radicals?

60. What is the principal root of $\pm\sqrt{4}$, $\sqrt[3]{8}$, and $\sqrt[3]{-8}$?

61. Name the order of $\sqrt{6}$; of $a^{\frac{1}{3}}$; of $\sqrt[3]{5}$; of $c^{\frac{2}{3}}$; of $\sqrt[4]{m^3}$.

62. Give an example of (*a*) a real number; (*b*) an imaginary number; (*c*) a rational number; (*d*) an irrational number; (*e*) a radical; (*f*) a surd; (*g*) an index; (*h*) a radicand; (*i*) the principal odd root of a positive number; (*j*) the principal even root of a positive number; (*k*) the principal odd root of a negative number.

70. Simplification of radicals. The form of a radical expression may be changed without altering its numerical value. It is often desirable to change the form of a radical

so that its numerical value can be computed with the least possible labor.

The simplification of a radical is based on the general identity

$$\sqrt[n]{a^nb} = \sqrt[n]{a^n} \cdot \sqrt[n]{b} = a\sqrt[n]{b}.$$

A radical is in its simplest form when the radicand

I. Is integral.

II. Contains no rational factor raised to a power which is equal to, or greater than, the order of the radical.

III. Is not raised to a power, unless the exponent of the power and the index of the root are prime to each other.

For the meaning of I, II, and III study carefully the

EXAMPLES

Examples of I:

1. $\sqrt{\frac{3}{2}} = \sqrt{\frac{6}{4}} = \sqrt{\frac{1}{4} \cdot 6} = \sqrt{\frac{1}{4}}\,\sqrt{6} = \frac{1}{2}\sqrt{6}.$

2. $6\sqrt{\frac{1}{3}} = 6\sqrt{\frac{3}{9}} = 6\sqrt{\frac{1}{9} \cdot 3} = 6 \cdot \frac{1}{3}\sqrt{3} = 2\sqrt{3}.$

3. $\sqrt{\frac{3}{5x}} = \sqrt{\frac{15x}{25x^2}} = \sqrt{\frac{1}{25x^2} \cdot 15x} = \frac{1}{5x}\sqrt{15x}.$

Examples of II:

1. $\sqrt{4x^5} = \sqrt{4x^4 \cdot x} = \sqrt{(2x^2)^2 \cdot x} = 2x^2\sqrt{x}.$

2. $5\sqrt[3]{24x^5} = 5\sqrt[3]{8x^3 \cdot 3x^2} = 5\sqrt[3]{(2x)^3 \cdot 3x^2} = 10x\sqrt[3]{3x^2}.$

3. $\sqrt{16 - 8\sqrt{2}} = \sqrt{4(4 - 2\sqrt{2})} = 2\sqrt{4 - 2\sqrt{2}}.$

Examples of III:

1. $\sqrt[4]{4} = \sqrt[4]{2^2} = 2^{\frac{2}{4}} = 2^{\frac{1}{2}} = \sqrt{2}.$

2. $\sqrt[6]{9} = \sqrt[6]{3^2} = 3^{\frac{2}{6}} = 3^{\frac{1}{3}} = \sqrt[3]{3}.$

3. $\sqrt[4]{a^2b^4} = a^{\frac{2}{4}}b^{\frac{4}{4}} = a^{\frac{1}{2}}b = b\sqrt{a}.$

A radical of the second order is simplified by the use of the

Rule. *Separate the radicand into two factors one of which is the greatest perfect square which it contains. Then take the square root of this factor and write it as the coefficient of a radical which has the other factor as radicand.*

If the original radical has a coefficient other than the number 1, multiply the result obtained above by this coefficient.

A similar rule holds for simplifying radicals involving the cube root and roots of higher orders.

EXERCISES

Simplify:

1. $\sqrt{18}$.	**6.** $\sqrt{52}$.	**11.** $\sqrt{192}$.	**16.** $3\sqrt[5]{64}$.
2. $\sqrt{20}$.	**7.** $\sqrt{63}$.	**12.** $2\sqrt{45}$.	**17.** $\sqrt{a^3}$.
3. $\sqrt{28}$.	**8.** $\sqrt{68}$.	**13.** $\sqrt[3]{16}$.	**18.** $\sqrt{ax^3}$.
4. $\sqrt{44}$.	**9.** $\sqrt{75}$.	**14.** $4\sqrt[3]{54}$.	**19.** $\sqrt[3]{a^2x^5}$.
5. $\sqrt{50}$.	**10.** $\sqrt{108}$.	**15.** $\sqrt[4]{48}$.	**20.** $5\sqrt[3]{16a^4}$.

21. $\sqrt{\frac{1}{3}}$.

Solution. $\sqrt{\frac{1}{3}} = \sqrt{\frac{3}{9}} = \sqrt{\frac{1}{9}\cdot 3} = \frac{1}{3}\sqrt{3}$.

22. $\sqrt{\frac{2}{5}}$. **23.** $\sqrt{\frac{3}{7}}$. **24.** $\sqrt[3]{\frac{1}{4}}$. **25.** $\sqrt[3]{\frac{2}{9}}$.

26. $\sqrt{\dfrac{1}{a}}$.

Solution. $\sqrt{\dfrac{1}{a}} = \sqrt{\dfrac{a}{a^2}} = \sqrt{\dfrac{1}{a^2}\cdot a} = \dfrac{1}{a}\sqrt{a}$.

27. $\sqrt{\dfrac{1}{2x}}$. **29.** $\sqrt[3]{\dfrac{1}{a^2}}$. **31.** $\sqrt[3]{\dfrac{1}{ax^3}}$. **34.** $6\sqrt[3]{-\frac{1}{9}}$.

35. $\sqrt{1-(\frac{1}{3})^2}$.

28. $\sqrt{\dfrac{1}{a^3}}$. **30.** $\sqrt[3]{\dfrac{2x}{a}}$. **32.** $\sqrt{\frac{5}{8}}$. **36.** $\sqrt{3^2-(\frac{3}{2})^2}$.

33. $\sqrt[3]{-\frac{3}{4}}$. **37.** $\sqrt{7^2+(\frac{7}{2})^2}$.

38. $\sqrt{s^2-\left(\frac{s}{2}\right)^2}$.

39. $\sqrt{s^3+\left(\frac{s}{2}\right)^3}$.

40. $\sqrt{\left(\frac{a+1}{2}\right)^2-a}$.

41. $\sqrt{\left(\frac{e^x+e^{-x}}{2}\right)^2-1}$.

42. $\sqrt{4-8\sqrt{3}}$.

HINT. $\sqrt{4-8\sqrt{3}}=\sqrt{4(1-2\sqrt{3})}$.

43. $\sqrt{36+18\sqrt{5}}$.

44. $\sqrt{a^3+a^2\sqrt{3}}$.

45. $\sqrt[3]{16-8\sqrt{3}}$.

46. $\sqrt[3]{54-9\sqrt{18}}$.

47. $\sqrt{R^2+3R^2\sqrt{5}}$.

48. $\sqrt{\frac{R^2+R^2\sqrt{6}}{3}}$.

49. $\sqrt{R^2-\frac{R^2}{2}\sqrt{3}}$.

50. $\sqrt[4]{4}$.

Solution. $\sqrt[4]{4}=\sqrt[4]{2^2}$.
$2^{\frac{2}{4}}=2^{\frac{1}{2}}=\sqrt{2}$.

51. $\sqrt[4]{9}$.

52. $\sqrt[6]{4}$.

53. $\sqrt[4]{a^2b^4}$.

54. $\sqrt[4]{4a^6}$.

55. $\sqrt[4]{64a^2b^4}$.

56. $\sqrt[6]{9a^2}$.

57. $\sqrt[4]{\frac{a^2}{9}}$.

58. $\sqrt[6]{\frac{4a^6}{25}}$.

Express entirely under the radical sign:

59. $2\sqrt{7}$.

Solution. $2\sqrt{7}=\sqrt{4}\sqrt{7}=\sqrt{28}$.

60. $3\sqrt{5}$.

61. $4\sqrt{3}$.

62. $2\sqrt[3]{8}$.

63. $a\sqrt{a}$.

64. $2c\sqrt[3]{c^2}$.

65. $4\sqrt[3]{\frac{1}{4}}$.

66. $x^3\sqrt[3]{x^2}$.

67. $\frac{a}{3}\sqrt[3]{\frac{9}{a^2}}$.

68. $e^x\sqrt{e^x+e^{-x}}$.

69. $(a+1)\sqrt{\frac{1}{a^2-1}}$.

70. $(2a+1)\sqrt{\frac{2}{4a^2-1}}$.

71. $\frac{x-3a}{5}\sqrt[3]{\frac{125}{(x-3a)^2}}$.

Express in simplest form with one radical sign:

72. $\sqrt{\sqrt{2}}$.

Solution. $\sqrt{\sqrt{2}}=\sqrt{2^{\frac{1}{2}}}$
$=2^{\frac{1}{4}}=\sqrt[4]{2}$.

73. $\sqrt{\sqrt{a}}$.

74. $\sqrt[3]{\sqrt{a}}$.

75. $\sqrt{\sqrt[3]{a^2}}$.

76. $\sqrt{\sqrt{x^3}}$.

77. $\sqrt{\sqrt[3]{x}}$.

78. $\sqrt[3]{\sqrt[3]{x}}$.

79. $\sqrt[3]{\sqrt{x^5}}$.

80. $\sqrt[3]{\sqrt{8a^2x}}$.

81. $\sqrt{3\sqrt{3}}$.

82. $\sqrt{3\sqrt{3\sqrt{3}}}$.

83. $\sqrt[4]{\sqrt{8}}$.

84. $2\sqrt[3]{2\sqrt[3]{2}}$.

85. $\sqrt{\sqrt{\sqrt{x^{12}}}}$.

86. $\sqrt[a]{\sqrt[n]{x^c}}$.

87. $\sqrt[\frac{m}{n}]{x^{\frac{a}{n}}}$.

Find by the formula of Exercise 28, page 108, the areas of the triangles whose sides are

88. 6, 8, and 10.

89. 7, 24, and 25.

90. 33, 56, and 65.

91. 104, 153, and 185.

71. Addition and subtraction of radicals. Similar radicals are radicals of the same order, with radicands which are identical or which can be made so by simplification.

The sum or the difference of similar radicals can be expressed as one term, while the sum or difference of dissimilar radicals can only be indicated.

EXERCISES

Simplify and collect:

1. $\sqrt{8}+\sqrt{18}$.

Solution. $\sqrt{8}+\sqrt{18}=$
$2\sqrt{2}+3\sqrt{2}=5\sqrt{2}$.

2. $\sqrt{\frac{1}{2}}+3\sqrt{2}$.

3. $\sqrt{50}+\sqrt{98}-\sqrt{32}$.

4. $\sqrt{12}+5\sqrt{75}-2\sqrt{27}$.

5. $3\sqrt{18}-\sqrt{98}+\sqrt{128}$.

6. $\sqrt{75}+3\sqrt{147}-\sqrt{12}$.

7. $2\sqrt{54}+\sqrt{24}-\sqrt{96}$.

8. $\sqrt{45}-\sqrt{20}+5\sqrt{245}$.

9. $3\sqrt{275}+2\sqrt{99}-5\sqrt{44}$.

10. $\sqrt[3]{16}+\sqrt[3]{54}-3\sqrt[3]{2}$.

11. $\sqrt[3]{192}-4\sqrt[3]{24}+\sqrt[3]{375}$.

12. $\sqrt[3]{54}+\sqrt[3]{16}-\sqrt[3]{128}$.

13. $\sqrt[3]{625}+\sqrt[3]{40}+\sqrt[3]{135}$.

14. $10\sqrt{\frac{6}{5}}-\sqrt{\frac{3}{10}}+\sqrt{\frac{15}{2}}$.

15. $3\sqrt{\frac{2}{7}}+3\sqrt{\frac{7}{2}}-2\sqrt{\frac{1}{14}}$.

16. $a\sqrt{x^3}-\sqrt{a^2x}-5\sqrt{a^2x}$.

17. $\sqrt{x^3}+\sqrt[4]{x^2}-12\sqrt[6]{x^3}$.

18. $\sqrt{\dfrac{3a}{x}}+\sqrt{\dfrac{3x}{a}}-\sqrt{\dfrac{ax}{3}}$.

19. $\sqrt{\dfrac{a}{x^3}}-\sqrt{\dfrac{a}{x^5}}+\sqrt{\dfrac{5x^3}{a}}$.

20. $\sqrt{\frac{6}{7}}+\sqrt{\frac{3}{14}}-\sqrt{\frac{21}{2}}$.

21. $\sqrt[4]{32x^5}+\sqrt[4]{1250x}-\sqrt[4]{512x}-\sqrt[4]{2x}.$

22. $\sqrt{(a+c)^3}-c\sqrt[4]{(a+c)^2}+2c\sqrt[6]{(a+c)^3}.$

23. $\sqrt[3]{(a-c)^4}+c\sqrt[6]{a^2-2ac+c^2}+(a+c)\sqrt[3]{a-c}.$

24. $\sqrt{\frac{a}{c}}-\sqrt{\frac{c}{a}}+\sqrt{\frac{a^2+c^2}{ac}+2}-\sqrt{\frac{a^2+c^2}{ac}-2}.$

25. $\sqrt[3]{24}+\sqrt[3]{(3a+9)(a+3)^2}-\sqrt[3]{81}+a\sqrt[6]{9}-4\sqrt[3]{3}.$

26. $2\sqrt{9a^3-9a^2b}-3\sqrt{9ab^2-9b^3}+\sqrt{(a^2-b^2)(a+b)}.$

27. $(a-b)\sqrt{\frac{a+b}{a-b}}+\sqrt{25a^2-25b^2}+\frac{a+b}{a-b}\sqrt{\frac{36ab^2-36b^3}{a+b}}.$

72. Multiplication of real radicals. Real radicals of the same order are multiplied as follows:

Example 1. Multiply $2\sqrt{x}-3\sqrt{a}-4\sqrt{ax}$ by $2\sqrt{ax}$.

Solution.

$$\begin{array}{l} 2\sqrt{x}-3\sqrt{a}-4\sqrt{ax} \\ 2\sqrt{ax} \\ \hline 4x\sqrt{a}-6a\sqrt{x}-8ax \end{array}$$

Real radicals of different order are multiplied as follows:

Example 2. Multiply $\sqrt{n}$ by $\sqrt[3]{x}$.

Solution.

$$\sqrt{n}=n^{\frac{1}{2}}=n^{\frac{3}{6}}=\sqrt[6]{n^3}.$$

$$\sqrt[3]{x}=x^{\frac{1}{3}}=x^{\frac{2}{6}}=\sqrt[6]{x^2}.$$

Then $$\sqrt{n}\cdot\sqrt[3]{x}=\sqrt[6]{n^3}\cdot\sqrt[6]{x^2}=\sqrt[6]{n^3x^2}.$$

The method of multiplying real radicals is stated in the

Rule. *If necessary, reduce the radicals to the same order.*

Find the products of the coefficients of the radicals for the coefficient of the radical part of the result.

Multiply together the radicands and write the product under the common radical sign.

Reduce the result to its simplest form.

The preceding rule does not hold for the multiplication of imaginary numbers. This case is discussed in Chapter XII.

EXERCISES

Perform the indicated multiplication and simplify the products:

1. $\sqrt{3}\sqrt{27}$.
2. $\sqrt{12}\sqrt{18}$.
3. $\sqrt{\frac{2}{3}}\cdot\sqrt{\frac{3}{4}}$.
4. $\sqrt{\frac{4}{5}}\cdot\sqrt{\frac{5}{6}}\cdot\sqrt{\frac{3}{4}}$.
5. $\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{3}}\cdot\sqrt{\frac{1}{5}}$.
6. $(\sqrt{3}-\sqrt{a})\sqrt{2}$.
7. $(\sqrt{2}-3\sqrt{5})\sqrt{5}$.
8. $(\sqrt{3}-2\sqrt{2})(\sqrt{2}-\sqrt{3})$.
9. $(\sqrt{5}-3\sqrt{2})(2\sqrt{5}-\sqrt{3})$.
10. $(\sqrt{a}-\sqrt{ax})(\sqrt{a}+2\sqrt{ax})$.
11. $(\sqrt{2}+\sqrt{3})(\sqrt{2}-\sqrt{3})$.
12. $(3\sqrt{5}-\sqrt{2})(3\sqrt{5}+\sqrt{2})$.
13. $(\sqrt{7}-\sqrt{5})(\sqrt{7}+\sqrt{5})$.
14. $(2\sqrt{3}-\sqrt{3})(2\sqrt{3}+\sqrt{3})$.
15. $(4\sqrt{5}+2\sqrt{7})(4\sqrt{5}-2\sqrt{7})$.
16. $(\sqrt{3a}-\sqrt{2a})^2$.
17. $(\sqrt{x-3})^2$.
18. $(2\sqrt{3x-1})^2$.
19. $(\sqrt{x}-\sqrt{x-2})^2$.
20. $3\sqrt{x-3}\sqrt{4x-8}$.
21. $(\sqrt{5}-\sqrt{3}-\sqrt{2})(\sqrt{5}+\sqrt{3}-\sqrt{2})$.
22. $(3\sqrt{2}+2\sqrt{3}+\sqrt{30})(2\sqrt{2}+2\sqrt{3}-2\sqrt{5})$.
23. $\left(R-\frac{R}{2}\sqrt{3}\right)\left(2R+\frac{3R}{2}\sqrt{3}\right)$.
24. $\left(\frac{R}{2}\right)^2+\left(\frac{R}{2}-\frac{R}{2}\sqrt{3}\right)^2$.
25. $R^2-\left(\frac{R}{2}\sqrt{5}-\frac{R}{2}\right)^2$.

Square:

26. $\sqrt{2}-\sqrt{x-3}$.
27. $\sqrt{x}-\sqrt{x+4}$.
28. $\sqrt{x-3}+\sqrt{x+5}$.
29. $\sqrt{x-3}-\sqrt{x+3}$.
30. $2\sqrt{x}-3\sqrt{2x+1}$.
31. $3\sqrt{x-1}+2\sqrt{2-x}$.

Perform the indicated multiplication:

32. $(a+\sqrt{a+b})(a-\sqrt{a+b})$.

33. $(\sqrt{a-b}-\sqrt{a})(\sqrt{a-b}+\sqrt{a})$.

34. $(\sqrt{2x-3}-\sqrt{3x})(\sqrt{2x-3}+\sqrt{3x})$.

Express as radicals of same order:

35. $\sqrt{2}$ and $\sqrt[3]{3}$.

36. $\sqrt{a}$ and $\sqrt[3]{a^2}$.

37. $\sqrt[4]{ax}$ and $\sqrt{a}$.

38. $\sqrt[3]{9}$ and $\sqrt{8}$.

Multiply the following:

39. $\sqrt{3}, \sqrt[3]{3}$.

40. $\sqrt{a}, \sqrt[3]{a}$.

41. $\sqrt{8}, \sqrt[3]{8}$.

42. $\sqrt[4]{a^3}, \sqrt{a}$.

43. $\sqrt[4]{6}, \sqrt{2}$.

44. $\sqrt[3]{12}, \sqrt{\frac{1}{8}}$.

45. $\sqrt[3]{c}, \sqrt{a}$.

46. $\sqrt[3]{a^2}, \sqrt{a^3}$.

47. $\sqrt{ax}, \sqrt[3]{a^2x}$.

48. $\sqrt{\dfrac{x}{a}}, \sqrt[3]{\dfrac{a}{x}}$.

49. $\sqrt[4]{2x^3}, \sqrt{3x}$.

50. $\sqrt{a+b}, \sqrt[3]{a+b}$.

73. Division of radicals. Direct division of radicals coefficient by coefficient and radicand by radicand is often possible.

Thus $$6\sqrt{5}\div 3\sqrt{3}=2\sqrt{\tfrac{5}{3}}=\tfrac{2}{3}\sqrt{15},$$

and $$3\sqrt{ax}\div 2\sqrt{x}=\tfrac{3}{2}\sqrt{a}.$$

Direct division of radicals when the divisor is a radical expression with more than one term is usually very difficult. In such cases a rationalizing factor of the denominator is used. We then carry out the operation of division indirectly by resorting to multiplication.

74. Rationalizing factor. One radical expression is a **rationalizing factor** for another if the product of the two is rational.

A rationalizing factor for $\sqrt{7}$ is $\sqrt{7}$, since $\sqrt{7}\cdot\sqrt{7}=7$. For $\sqrt[3]{5}$ a rationalizing factor is $\sqrt[3]{25}$, since $\sqrt[3]{5}\cdot\sqrt[3]{25}=5$. Similarly, $\sqrt{5}-\sqrt{3}$ is a rationalizing factor of $\sqrt{5}+\sqrt{3}$, as their product, $(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})$, is equal to $5-3$, or 2.

In like manner $(3\sqrt{7}+2\sqrt{5})(3\sqrt{7}-2\sqrt{5})=63-20=43$.

Two important radical expressions are $\sqrt{a}+\sqrt{b}$ and $\sqrt{a}-\sqrt{b}$. Two such binomials are called **conjugate** radicals, and either is a rationalizing factor for the other.

Rationalizing factors are used in division of radicals as follows:

Example 1. $\sqrt{6}\div\sqrt{5}=\dfrac{\sqrt{6}\sqrt{5}}{\sqrt{5}\sqrt{5}}=\dfrac{\sqrt{30}}{5}.$

Example 2.
$$(6\sqrt{2}-15\sqrt{3})\div 3\sqrt{5}=\frac{(6\sqrt{2}-15\sqrt{3})\cdot\sqrt{5}}{3\sqrt{5}\cdot\sqrt{5}}=\frac{2\sqrt{10}-5\sqrt{15}}{5}.$$

Example 3.
$$(\sqrt{3}+\sqrt{2})\div(\sqrt{5}-\sqrt{3})=\frac{(\sqrt{3}+\sqrt{2})(\sqrt{5}+\sqrt{3})}{(\sqrt{5}-\sqrt{3})(\sqrt{5}+\sqrt{3})}=\frac{\sqrt{15}+\sqrt{10}+3+\sqrt{6}}{5-3}=\tfrac{1}{2}(\sqrt{15}+\sqrt{10}+3+\sqrt{6}).$$

Therefore when direct division of radicals is impossible use the

Rule. *Write the dividend over the divisor in the form of a fraction. Then multiply the numerator and denominator of the fraction by a rationalizing factor for the denominator and simplify the resulting fraction.*

This rule applies in all cases, while the rule for direct division fails when dividing a real radical by an imaginary number.

EXERCISES

Find a simple rationalizing factor for:

1. $\sqrt{7}$.
2. $3\sqrt{5}$.
3. $\sqrt{8}$.
4. $\sqrt[3]{4}$.
5. $\sqrt[3]{5}$.
6. $\sqrt[4]{8}$.
7. $\sqrt{5}-\sqrt{7}$.
8. $\sqrt{3}-2$.
9. $3\sqrt{6}-2\sqrt{11}$.
10. $\sqrt{3a}-\sqrt{2x}$.
11. $\sqrt{x-3}-\sqrt{3}$.
12. $\sqrt{x-3}-2\sqrt{3x}$.
13. $\sqrt{a-b}+\sqrt{a+b}$.
14. $\sqrt{8}+\sqrt{2}-\sqrt{5}$.
15. $\sqrt{3}+\sqrt{18}-\sqrt{2}$.

Perform the indicated division:

16. $\sqrt{8}\div\sqrt{2}$.
17. $6\sqrt{10}\div\sqrt{5}$.
18. $\sqrt{12}\div\sqrt{3}$.
19. $\sqrt{8}\div\sqrt{24}$.
20. $\sqrt{a^3x}\div\sqrt{ax^2}$.
21. $\sqrt{2ax}\div\sqrt{3a^2x}$.
22. $(\sqrt{14}-\sqrt{10})\div\sqrt{2}$.
23. $(2\sqrt{10}-3\sqrt{5})\div 2\sqrt{5}$.
24. $(\sqrt{ax^2}-\sqrt{a^2x})\div\sqrt{ax}$.
25. $8\div 4\sqrt{3}$.
26. $8\div 2\sqrt{3}$.
27. $24\div 3\sqrt{3}$.
28. $\sqrt{3}\div\sqrt[3]{2}$.

Solution. $\dfrac{\sqrt{3}}{\sqrt[3]{2}}=\dfrac{\sqrt{3}\cdot\sqrt[3]{4}}{\sqrt[3]{2}\cdot\sqrt[3]{4}}=\dfrac{\sqrt{3}\cdot\sqrt[3]{4}}{2}=\dfrac{\sqrt[6]{27\cdot 16}}{2}$.

29. $\sqrt{5}\div\sqrt[3]{3}$.
30. $\sqrt{a}\div\sqrt[4]{2}$.
31. $\sqrt[4]{8}\div\sqrt{2}$.
32. $\sqrt[3]{a}\div\sqrt[6]{a}$.
33. $\sqrt[3]{2}\div\sqrt[6]{x^2}$.
34. $\sqrt{32}\div\sqrt[4]{2}$.
35. $\sqrt[3]{\frac{1}{4}}\div\sqrt{\frac{1}{2}}$.
36. $a\div c\sqrt{x}$.
37. $\sqrt{3}\div(\sqrt{3}-2)$.
38. $\sqrt{5}\div(\sqrt{5}+\sqrt{2})$.
39. $(2\sqrt{3}+\sqrt{5})\div(\sqrt{3}-\sqrt{5})$
40. $(\sqrt{7}+\sqrt{3})\div(\sqrt{7}-\sqrt{2})$.

Change to respectively equivalent fractions having rational denominators:

41. $\dfrac{\sqrt{5}}{\sqrt{5}-\sqrt{3}}$. **44.** $\dfrac{2\sqrt{5}+3\sqrt{7}}{3\sqrt{5}-2\sqrt{7}}$. **47.** $\dfrac{\sqrt{x-3}+\sqrt{3}}{\sqrt{x-3}-\sqrt{3}}$.

42. $\dfrac{\sqrt{3}}{2\sqrt{6}-\sqrt{3}}$. **45.** $\dfrac{\sqrt{a}+\sqrt{b}}{\sqrt{a}-\sqrt{b}}$. **48.** $\dfrac{\sqrt{a+x}}{\sqrt{5}-\sqrt{a+x}}$.

43. $\dfrac{\sqrt{5}+\sqrt{2}}{\sqrt{5}-\sqrt{2}}$. **46.** $\dfrac{\sqrt{x}-2\sqrt{c}}{\sqrt{x}+\sqrt{c}}$. **49.** $\dfrac{4}{\sqrt{2-\sqrt{2}}}$.

Perform the indicated division:

50. $(\sqrt{10}-\sqrt{5})\div(\sqrt{10}+\sqrt{5})$.

51. $(x-\sqrt{c})\div(x-3\sqrt{c})$.

52. $(\sqrt{a+c}-\sqrt{x})\div(\sqrt{a+c}+\sqrt{x})$.

53. $(2-\sqrt{3}+\sqrt{2})\div(\sqrt{3}+\sqrt{2})$.

54. $(\sqrt{5}+\sqrt{7}-\sqrt{2})\div(\sqrt{5}-\sqrt{7})$.

55. Is there any distinction between the meaning of the direction before Exercise 41 and of that before Exercise 50?

56. Does $3-\sqrt{7}$ satisfy $x^2-6x+2=0$?

57. Does $\dfrac{7-\sqrt{3}}{2}$ satisfy $2x^2-75x+161=0$?

58. Does $\frac{1}{6}(5\pm\sqrt{109})$ satisfy $3x^2-5x-7=0$?

75. Square root of surd expressions. The square of a binomial is usually a trinomial. However, the result of squaring a binomial of the form $\sqrt{a}+\sqrt{b}$ is a binomial if a and b are rational numbers. Thus

$$(\sqrt{7}-\sqrt{3})^2=7-2\sqrt{21}+3=10-2\sqrt{21}.$$

In $10-2\sqrt{21}$, 10 is the sum of 7 and 3, and 21 is the product of 7 and 3. These relations and the fact that the coefficient of the radical $\sqrt{21}$ is 2 enable us to find

the square root of many expressions of the form $a \pm \sqrt{b}$ by writing each in the form of $x \pm 2\sqrt{xy} + y$ and then extracting the square root of the trinomial square as follows:

Example. Extract the square root of $9 - \sqrt{56}$.

Solution. $9 - \sqrt{56} = 9 - 2\sqrt{14}$.

We must now find two numbers whose sum is 9 and whose product is 14. These are 7 and 2.

Then $9 - 2\sqrt{14} = 7 - 2\sqrt{14} + 2 = (\sqrt{7} - \sqrt{2})^2$.

Hence the square roots of $9 - \sqrt{56}$ are $\pm (\sqrt{7} - \sqrt{2})$.

EXERCISES

Find the positive square roots in Exercises 1–13:

1. $8 - 2\sqrt{15}$.
2. $5 - 2\sqrt{6}$.
3. $13 + \sqrt{48}$.
4. $7 - \sqrt{40}$.
5. $16 - 6\sqrt{7}$.
6. $17 + 12\sqrt{2}$.
7. $11 - 3\sqrt{8}$.
8. $11 - \sqrt{120}$.
9. $65x - 20\sqrt{3x^2}$.
10. $126a - 10a\sqrt{5}$.
11. $\dfrac{13a}{4} - \sqrt{3a^2}$.
12. $2x + 2\sqrt{x^2 - 49}$.
13. $a + \sqrt{a^2 - 1}$.
14. $\sqrt{9 + 3\sqrt{8}} = \sqrt{?} + \sqrt{?}$.
15. $\sqrt{15 - 5\sqrt{8}} = ?$
16. $\sqrt{a + \sqrt{a^2 - 4b^2}} = ?$
17. $\sqrt{m^2 + m + 2n + 2m\sqrt{m + 2n}} = ?$

NOTE. In the writings of one of the later Hindu mathematicians (about A.D. 1150) we find a method of extracting the square root of surds which is practically the same as that given in the text. In fact, the formula for the operation is given, apart from the modern symbols, as follows: $\sqrt{a} + \sqrt{b} = \sqrt{a + b + 2\sqrt{ab}}$. The study of expressions of the type $\sqrt{\sqrt{a} \pm \sqrt{b}}$ had been carried to a most remarkable degree of accuracy by the Greek, Euclid. His researches on this subject, if original with him, place him among the keenest mathematicians of all time, but his work and all of his results are expressed in geometrical language which is very far removed from the algebraic symbolism of to-day.

76. Factors involving radicals. In the chapter on Factoring it was definitely stated that factors involving radicals would not then be considered. This limitation on the character of a factor is no longer necessary. Consequently many expressions which previously have been regarded as prime may now be thought of as factorable.

Thus $$3x^2-1=(x\sqrt{3}+1)(x\sqrt{3}-1),$$
and $$4x^2-5=(2x+\sqrt{5})(2x-\sqrt{5}).$$

In this extension of our notion of a factor it must be clearly understood that the use of radicals is limited to the coefficients in the terms of the factors.

To restrict the use of radicals in the way just indicated is necessary for the sake of definiteness. Otherwise it would be impossible to obey a direction to factor even so simple an expression as x^2-y^2; for if the unknown is allowed under a radical sign in a factor, x^2-y^2 has countless factors.

Thus $$\begin{aligned} x^2-y^2 &= (x+y)(x-y) \\ &= (x+y)(\sqrt{x}+\sqrt{y})(\sqrt{x}-\sqrt{y}) \\ &= (x+y)(\sqrt{x}+\sqrt{y})(\sqrt[4]{x}+\sqrt[4]{y})(\sqrt[4]{x}-\sqrt[4]{y}), \end{aligned}$$
and so on indefinitely.

EXERCISES

Factor:

1. x^2-11.
2. $3x^2-8$.
3. x^3+3.
4. x^3-12.
5. $3x^3-27$.
6. $5x^3+125$.

Find the algebraic sum of the following:

7. $\dfrac{2\sqrt{b}}{a-b}+\dfrac{2}{\sqrt{a}+\sqrt{b}}$.
8. $\dfrac{x+c}{\sqrt{x}-\sqrt{c}}-\dfrac{x^{\frac{3}{2}}+c^{\frac{3}{2}}}{x-c}$.

Solve by factoring, and check the results:

9. $x^2-5=0$.
10. $2x^2-3=0$.
11. $x^4+144=26x^2$.
12. $4x^4+c=x^2+4cx^2$.

PROBLEMS

(Obtain answers in simplest radical form.)

1. The side of an equilateral triangle is 8. Find the altitude.

2. The side of an equilateral triangle is s. Find the altitude and the area.

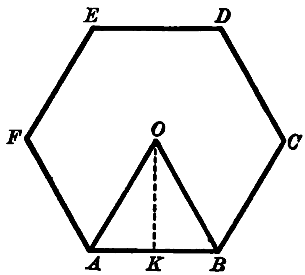
3. The altitude of an equilateral triangle is 24. Find one side and the area.

4. Find the side of an equilateral triangle whose altitude is a.

5. Find the altitude on the longest side of the triangle whose sides are 11, 13, and 20. Find the area of the triangle.

HINT. Let the altitude on the side 20 be x, and the two parts into which the altitude divides side 20 be y and $20 - y$; then set up two equations involving x and y, and solve.

6. Find the altitude on the longest side of the triangle whose sides are 10, 12, and 16.



Fact from Geometry. A regular hexagon may be divided into six equal equilateral triangles by lines from its center to the vertices.

In the adjacent regular hexagon, $AB = BC = CD$, etc. O is the center and OK perpendicular to AB is the **apothem** of the hexagon.

7. Find the apothem and the area of a regular hexagon (*a*) whose side is 18; (*b*) whose side is s.

8. Find the side and the area of a regular hexagon (*a*) whose apothem is 30; (*b*) whose apothem is h.

Facts from Geometry. The volume of a pyramid or cone is $\frac{ab}{3}$, where a is the altitude and b is the area of the base.

The altitudes of an equilateral triangle intersect at a point which divides each altitude into two parts whose ratio is 2 to 1.

79. $\sqrt[3]{\sqrt{x^5}}$.
80. $\sqrt[3]{\sqrt{8a^2x}}$.
81. $\sqrt{3\sqrt{3}}$.
82. $\sqrt{3\sqrt{3\sqrt{3}}}$.
83. $\sqrt[4]{\sqrt{8}}$.
84. $2\sqrt[3]{2\sqrt[3]{2}}$.
85. $\sqrt{\sqrt{\sqrt{x^{12}}}}$.
86. $\sqrt[a]{\sqrt[n]{x^c}}$.
87. $\sqrt[\frac{m}{n}]{\frac{a}{x^n}}$.

Find by the formula of Exercise 28, page 108, the areas of the triangles whose sides are

88. 6, 8, and 10.
89. 7, 24, and 25.
90. 33, 56, and 65.
91. 104, 153, and 185.

71. Addition and subtraction of radicals. Similar radicals are radicals of the same order, with radicands which are identical or which can be made so by simplification.

The sum or the difference of similar radicals can be expressed as one term, while the sum or difference of dissimilar radicals can only be indicated.

EXERCISES

Simplify and collect:

1. $\sqrt{8}+\sqrt{18}$.

Solution. $\sqrt{8}+\sqrt{18}=$
$2\sqrt{2}+3\sqrt{2}=5\sqrt{2}$.

2. $\sqrt{\frac{1}{2}}+3\sqrt{2}$.
3. $\sqrt{50}+\sqrt{98}-\sqrt{32}$.
4. $\sqrt{12}+5\sqrt{75}-2\sqrt{27}$.
5. $3\sqrt{18}-\sqrt{98}+\sqrt{128}$.
6. $\sqrt{75}+3\sqrt{147}-\sqrt{12}$.
7. $2\sqrt{54}+\sqrt{24}-\sqrt{96}$.
8. $\sqrt{45}-\sqrt{20}+5\sqrt{245}$.
9. $3\sqrt{275}+2\sqrt{99}-5\sqrt{44}$.
10. $\sqrt[3]{16}+\sqrt[3]{54}-3\sqrt[3]{2}$.
11. $\sqrt[3]{192}-4\sqrt[3]{24}+\sqrt[3]{375}$.
12. $\sqrt[3]{54}+\sqrt[3]{16}-\sqrt[3]{128}$.
13. $\sqrt[3]{625}+\sqrt[3]{40}+\sqrt[3]{135}$.
14. $10\sqrt{\frac{6}{5}}-\sqrt{\frac{3}{10}}+\sqrt{\frac{15}{2}}$.
15. $3\sqrt{\frac{2}{7}}+3\sqrt{\frac{7}{2}}-2\sqrt{\frac{1}{14}}$.
16. $a\sqrt{x^3}-\sqrt{a^2x}-5\sqrt{a^2x}$.
17. $\sqrt{x^3}+\sqrt[4]{x^2}-12\sqrt[6]{x^3}$.
18. $\sqrt{\frac{3a}{x}}+\sqrt{\frac{3x}{a}}-\sqrt{\frac{ax}{3}}$.
19. $\sqrt{\frac{a}{x^3}}-\sqrt{\frac{a}{x^5}}+\sqrt{\frac{5x^3}{a}}$.
20. $\sqrt{\frac{6}{7}}+\sqrt{\frac{3}{14}}-\sqrt{\frac{21}{2}}$.

21. $\sqrt[4]{32 x^5} + \sqrt[4]{1250 x} - \sqrt[4]{512 x} - \sqrt[4]{2 x}.$

22. $\sqrt{(a+c)^3} - c\sqrt[4]{(a+c)^2} + 2c\sqrt[6]{(a+c)^3}.$

23. $\sqrt[3]{(a-c)^4} + c\sqrt[6]{a^2 - 2ac + c^2} + (a+c)\sqrt[3]{a-c}.$

24. $\sqrt{\frac{a}{c}} - \sqrt{\frac{c}{a}} + \sqrt{\frac{a^2+c^2}{ac} + 2} - \sqrt{\frac{a^2+c^2}{ac} - 2}.$

25. $\sqrt[3]{24} + \sqrt[3]{(3a+9)(a+3)^2} - \sqrt[3]{81} + a\sqrt[6]{9} - 4\sqrt[3]{3}.$

26. $2\sqrt{9a^3 - 9a^2b} - 3\sqrt{9ab^2 - 9b^3} + \sqrt{(a^2-b^2)(a+b)}.$

27. $(a-b)\sqrt{\frac{a+b}{a-b}} + \sqrt{25a^2 - 25b^2} + \frac{a+b}{a-b}\sqrt{\frac{36ab^2 - 36b^3}{a+b}}.$

72. Multiplication of real radicals. Real radicals of the same order are multiplied as follows:

Example 1. Multiply $2\sqrt{x} - 3\sqrt{a} - 4\sqrt{ax}$ by $2\sqrt{ax}$.

Solution.

$$\begin{array}{l} 2\sqrt{x} - 3\sqrt{a} - 4\sqrt{ax} \\ 2\sqrt{ax} \\ \hline 4x\sqrt{a} - 6a\sqrt{x} - 8ax \end{array}$$

Real radicals of different order are multiplied as follows:

Example 2. Multiply $\sqrt{n}$ by $\sqrt[3]{x}$.

Solution.

$$\sqrt{n} = n^{\frac{1}{2}} = n^{\frac{3}{6}} = \sqrt[6]{n^3}.$$

$$\sqrt[3]{x} = x^{\frac{1}{3}} = x^{\frac{2}{6}} = \sqrt[6]{x^2}.$$

Then

$$\sqrt{n}\cdot\sqrt[3]{x} = \sqrt[6]{n^3}\cdot\sqrt[6]{x^2} = \sqrt[6]{n^3x^2}.$$

The method of multiplying real radicals is stated in the

Rule. *If necessary, reduce the radicals to the same order.*

Find the products of the coefficients of the radicals for the coefficient of the radical part of the result.

Multiply together the radicands and write the product under the common radical sign.

Reduce the result to its simplest form.

The preceding rule does not hold for the multiplication of imaginary numbers. This case is discussed in Chapter XII.

28. $$\frac{\dfrac{(x^2-1)ax^{a-1}-x^a\cdot 2x}{(x^2-1)^2}}{\dfrac{x^a}{x^2-1}}.$$

29. $$\frac{\dfrac{x^2(ax^{a-1})-(x^a+1)2x}{(x^2)^2}}{\dfrac{x^a+1}{x^2}}.$$

30. $$\frac{\dfrac{x^{2a}nx^{n-1}-x^n\cdot 2ax^{2a-1}}{(x^{2a})^2}}{\dfrac{x^n}{x^{2a}}}.$$

31. $$\frac{\dfrac{x^{-5}(-2x^{-1})-(x^2+3)(-5x^{-4})}{(x^{-5})^2}}{\dfrac{x^{-2}+3}{x^{-5}}}.$$

32. $$\frac{\dfrac{e^{-nx}(ne^{nx})-(e^{nx}+1)(-ne^{-nx})}{(e^{-nx})^2}}{\dfrac{2e^{nx}+1}{e^{-nx}}}.$$

33. $$\frac{\dfrac{(\sqrt[3]{x}+1)\frac{1}{3}x^{-\frac{2}{3}}-\sqrt[3]{x}(\frac{1}{3}x^{-\frac{2}{3}})}{(\sqrt[3]{x}+1)^2}}{\dfrac{\sqrt{x}}{\sqrt[3]{x}+1}}.$$

CHAPTER IX

FUNCTIONS AND THEIR GRAPHS

77. Functions. One of the most important concepts of mathematics is the notion of **function.**

As the term is used in mathematics the basic idea is the dependence of one quantity upon another or upon several others. Countless functional relations exist in everyday affairs. The velocity of a falling body is a function of the time since it started to fall, the interest on a definite sum of money is a function of the time and the rate, and the quantity of water transported yearly by the Mississippi is a function of the rainfall (and the snowfall) in its basin. Relations such as these can often be expressed either exactly or approximately by equations. For a body falling from rest near the earth's surface, $s = 16\,t^2$. Here the distance in feet, s, is expressed as a function of the time, t, in seconds. In $y = x^2 + 3\,x + 1$ *the value of y is a function of x.* Such a relation is often represented more clearly and strikingly by a graph of the equation than by the equation itself.

78. Names of functions. A function is called **linear, quadratic,** or **cubic** according as its degree with respect to the unknown or unknowns is first, second, or third respectively.

Thus $4\,x - 7$ is a linear function of x; $3\,x^2 - 8\,x + 1$ is a quadratic function of x; and $x^3 - 3\,x^2 + x - 10$ is a cubic function of x.

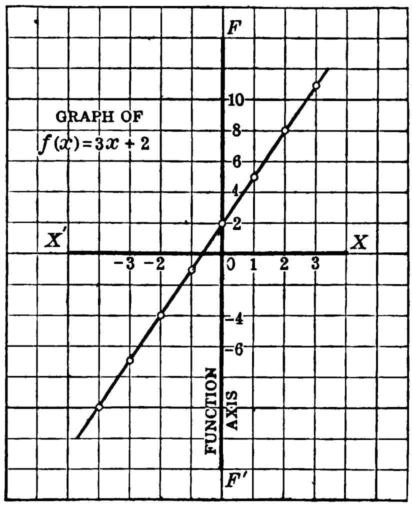
In the study of functions the unknown is often called the variable, since from this point of view the problem is not so much the finding of an unknown as it is the study of the changes of a variable quantity.

79. Notations for a function. After a function of any variable x has once been given it is usual to refer to it later in the same discussion by the symbol $f(x)$, which is read **the function of x**, or, more briefly, **f of x**.

80. Linear functions. The expression $3x + 2$ is a function of x, and the value of this binomial varies with x. The following table gives a partial view of the relative change of values between x and the function $3x + 2$:

If $x =$	-4	-3	-2	-1	0	1	2	3
then $f(x) = 3x + 2 =$	-10	-7	-4	-1	2	5	8	11

This relation can be represented graphically by using the same x-axis as before (section 44) and using the y-axis as the function axis; that is, laying off values of x horizontally and corresponding values of the function $3x + 2$ vertically. The graph resulting from the above table of values is shown in the accompanying figure. It can be shown that the graph of a linear function is always a **straight line.**



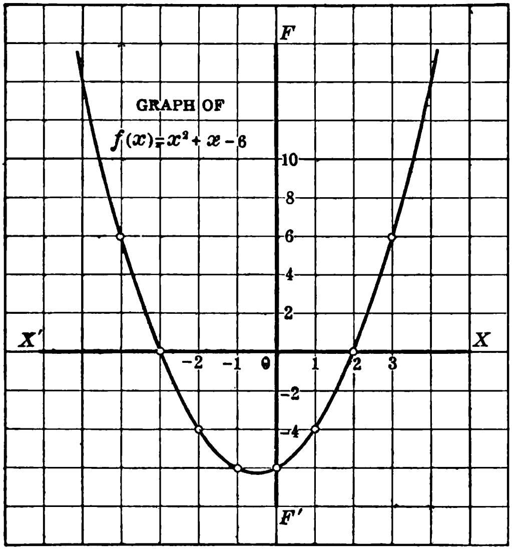
EXERCISES

1. Construct the graph of the function $3x - 1$.

2. Construct the graph of the function $2x + 8$.

81. Quadratic functions. The function $x^2 + x - 6$ may be represented graphically by proceeding as follows:

If $x =$	-4	-3	-2	-1	0	1	2	3
then $f(x) = x^2 + x - 6 =$	6	0	-4	-6	-6	-4	0	6



Plotting the points corresponding to the numbers in the table we obtain the accompanying graph.

The graph of a quadratic function in one variable is a curve called a **parabola.** It may be sharper or flatter than the accompanying graph, but of the same general shape, and the opening may be upward or downward.

EXERCISES

Construct the graph of the following:

1. $x^2 + 3x - 15.$ **3.** $x^2 + 4.$ **5.** $15 + 2x - x^2.$

2. $3x^2 + 5x + 2.$ **4.** $2x^2 - 5.$ **6.** $x^2 + 2x + 1.$

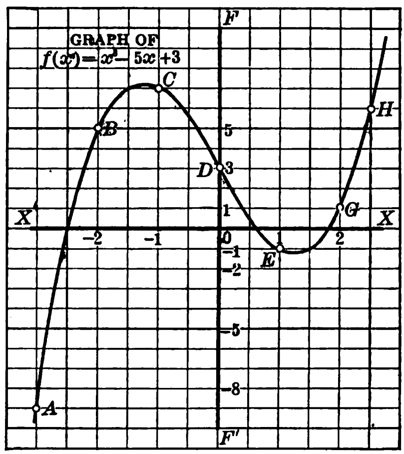
7. A body falling from rest near the earth's surface obeys the law $s = 16t^2$, where s is in feet and t is in seconds. Construct the graph of $f(t) = 16t^2$.

NOTE. In the study of analytical geometry one takes up systematically the curves which represent equations of the various degrees beginning with the simplest. It turns out, as we have already seen, that the linear equation is represented by a straight line. Equations of the second degree in x and y lead to the so-called "conic section."

One of the most interesting and important aspects of the graphical method is the fact that the simplest equations correspond to the most useful curves both in pure science and in nature. The commonest curves in nature are the circle and the parabola. Their equations are the very simplest equations of the second degree.

82. Graph of a cubic function. The graph of a cubic function is obtained in the same general way as that of a quadratic function. The function $x^3 - 5x + 3$ may be represented graphically by proceeding as follows:

If $x =$	-4	-3	-2	-1	0	1	2	$2\frac{1}{2}$	3
then $f(x) = x^3 - 5x + 3 =$	-41	-9	5	7	3	-1	1	$6\frac{1}{8}$	15



Plotting the points corresponding to the numbers in the table (except the first and last), we obtain the points $A(-3, -9)$, B, C, D, E, G, and H, in the order named. The curve crosses the x-axis three times: once between 1 and 2, again between 0 and 1, and a third time between -2 and -3. Above H the curve rises indefinitely, and below A it falls indefinitely. In each case it becomes more

and more nearly straight as it recedes from the x-axis, never crossing either axis again.

In forming a table of values two pairs are sufficient for a linear function, but more are needed for a quadratic function and still more for a cubic function. Usually the higher the degree of the function, the more points are needed in constructing its graph. It should be noted that in making a good graph the number of points is not so important as is their distribution, which should be such as faithfully to outline the entire curve. Where the graph curves rapidly or makes sharp turns the points should be close together. Such places are difficult to locate before the graph is constructed; hence one should make a table of values which appears to be sufficient and plot them. Then inspection of the plotted points will usually show where sharp turns or rapid curvature exists. The table of values should then be properly extended and the additional points located. Repetition of this last step will enable one to draw a graph which accurately pictures the variation of the function.

It should be observed here that scales on the two axes need not be the same. Some experience is required to choose for the two axes the scales which are best suited to bring out clearly the shape of the curve. In general the graph should be drawn to *as large* a scale, *in both directions*, as the size of the paper permits. What that will be for each axis can be decided by inspecting the table of values. For example, when the dimensions of the preceding graph are once determined one can see from the table that all values of x are easily represented but that it is undesirable to try to represent the values of the function, -41 and 15.

EXERCISES

Construct the graph of the following:

1. $x^3 - 3x + 1$.
2. $x^3 - 8x + 2$.
3. $x^3 - 4x - 2$.
4. $x^4 - 11x^2 + 24$.

NOTE. The notion of a function is one of the three or four most fundamental ideas in modern mathematics. Only the simplest examples are given in this book, but many others involving expressions of the utmost complexity have been studied by mathematicians for many years. An important reason for the study of functions is found

in the fact that all kinds of facts and principles which we meet in the study of nature can be expressed symbolically by means of functions, and the discovery of the properties of such functions helps us to understand the meaning of the facts. A complete understanding of the laws of falling bodies, light, electricity, or sound could never be reached without the study of the mathematical functions which these phenomena suggest.

When the electrician, the architect, or the artillerist meets a problem, he frequently must represent quantities by letters. The x and the y may represent the measures of objects in nature, but the solution has become merely an operation of algebra. As students of algebra we are not concerned with the origin of the function or expression, but merely with the numerical determination of some unknown or the simplification of some expression.

83. Graphical solution of equations in one unknown. In elementary algebra one of the most important applications of the preceding sections is their use in solving equations in one unknown. The ideas involved can be made clear by questions on the graphs of sections 80, 81, and 82.

ORAL EXERCISES

1. From the graph in section 80 determine the value of the function $3x + 2$ at the point where its graph crosses the x-axis.

2. Does this value of x satisfy the equation $3x + 2 = 0$?

3. Solve $3x + 2 = 0$ without reference to the graph.

4. What point on the graph represents the root of $3x + 2 = 0$?

5. From the graph in section 81 determine the values of the function $x^2 + x - 6$ at the points where its graph crosses the x-axis.

6. Do these values of x satisfy the equation $x^2 + x - 6 = 0$?

7. Solve $x^2 + x - 6 = 0$ without referring to the graph.

8. What points on the graph represent the roots of $x^2 + x - 6 = 0$?

9. From the graph in section 82 determine the values of the function $x^3 - 5x + 3$ at the points where its graph crosses the x-axis.

10. Do these values of x make the function $x^3 - 5x + 3$ equal zero?

11. What method could be used to solve the equation $x^3 - 5x + 3 = 0$?

84. The process of graphical solution. From what precedes, it is apparent that the steps in the graphical solution of an equation in one unknown are:

Transpose the terms so that the right member is zero.

Graph the function in the left member.

The values of x for the points where the graph crosses the x-axis are the real roots of the equation.

The algebraic solutions of a linear equation and a quadratic equation in one unknown are so simple that except for the purpose of illustration their graphical solution is comparatively unimportant. The algebraic solution of cubic and higher equations, except in simple cases, is much more difficult than the solution of the quadratic and is never presented in an elementary course. For this reason and for the insight it gives into equations in general, the *graphical* solution of the cubic and higher equations is important and illuminating. For such equations if the method of factoring fails, the graphical method is the only method open to the student at this point in his progress.

EXERCISES

Solve graphically:

1. $x^2 - 7x + 3 = 0.$

2. $x^3 - 5x + 4 = 0.$

3. $x^3 - 3x + 1 = 0.$

4. $x^3 - 8x = 0.$

5. $x^3 - x^2 + 5 = 0.$

6. $x^4 - 10x^2 + 16 = 0.$

85. Imaginary roots. An equation of the second or a higher degree often has **imaginary roots.** Such roots cannot be obtained by the graphical methods so far considered. A study of the graphs which follow will make clear why this is true.

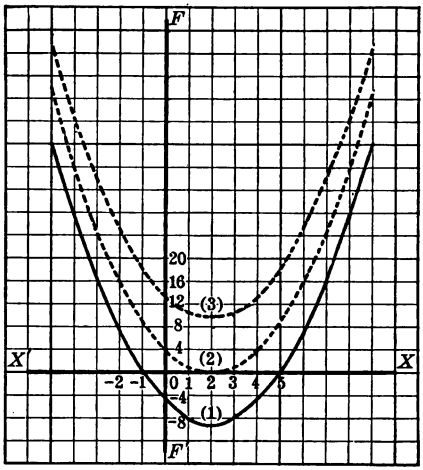
Consider the following equations:

$$x^2 - 4x - 5 = 0, \tag{1}$$

$$x^2 - 4x + 4 = 0, \tag{2}$$

$$x^2 - 4x + 13 = 0. \tag{3}$$

The graphs of the functions in the left members of equations (1), (2), and (3) are given in the accompanying figure. The three functions differ only in their constant terms, for 9 added to the constant term of (1) gives the constant term of (2), and 9 added to the constant term of (2) gives the constant term of (3). Apparently, as the constant term is increased the graph rises without change of shape and without motion to the left or to the right.

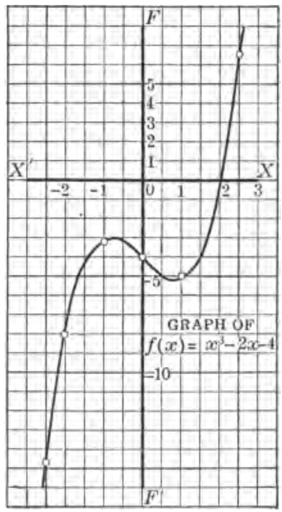


From the graph the roots of $x^2 - 4x - 5 = 0$ are seen to be 5 and -1. These results are obtained from factoring; $x^2 - 4x - 5 = 0$, or $(x-5)(x+1) = 0$. Whence $x = 5$ or -1.

If we imagine curve (1) to move upward, the two roots change in value and become the single root of curve (2),

which touches the x-axis at a point where x equals 2. Solving $x^2 - 4x + 4 = 0$ by factoring gives $(x-2)(x-2) = 0$. Whence $x = 2$.

If we now imagine curve (1) to move still farther upward from its position (2), it will no longer cut the x-axis. Further, when the curve reaches the position of (3) it does not cut the x-axis at all, and hence cannot show the values of the roots of the equation $x^2 - 4x + 13 = 0$, as, in fact, it does not. The graph does show, however, that the value of $x^2 - 4x + 13$ at the lowest point of the curve is 9. This means that for every real value of x, positive or negative, $x^2 - 4x + 13$ is never less than 9. The graph of (3) makes clear that no real number if substituted for x will make $x^2 - 4x + 13$ equal zero.



It can be shown by the method of section 87 that the roots of $x^2 - 4x + 13 = 0$ are $2 + 3\sqrt{-1}$ and $2 - 3\sqrt{-1}$.

NOTE. It required the genius of no less a man than Sir Isaac Newton first to observe from the graph of a function that two of its roots become imaginary simultaneously. He also saw that an equation with two of its roots equal to each other is, in a certain sense, the limiting case between equations in which the corresponding roots appear as two real and distinct roots and those in which they appear as imaginary roots.

86. Imaginary roots for a cubic equation. If we attempt to solve $x^3 - 2x - 4 = 0$ graphically, we obtain the graph of the accompanying figure. The curve crosses the x-axis at $x = 2$. This is the only real root the equation has; the other

two are imaginary. The roots can here be obtained by factoring $x^3 - 2x - 4 = 0$; thus $(x-2)(x^2+2x+2) = 0$. The roots of $x^2+2x-2=0$ are the two imaginary roots of the *cubic* equation.

EXERCISES

As far as possible solve graphically, finding results to one decimal place:

1. $x^2 - 5x - 9 = 0.$
2. $x^2 = 4x - 5.$
3. $x^3 - 3x + 4 = 0.$
4. $x^3 + x - 4 = 0.$
5. $x^3 - 4x^2 - 2x + 8 = 0.$
6. $x^3 - 4x + 5 = 0.$
7. $x^3 - 7x + 4 = 0.$
8. $x^4 - 4x^3 + 12 = 0.$
9. $x^4 = 10x^2 - 9.$
10. $x^4 - 2x^2 = 0.$

CHAPTER X

QUADRATIC EQUATIONS

87. Solution by completing the square. An equation of the form $ax^2 + bx + c = 0$, where a, b, and c denote numbers or known literal expressions, is called a **quadratic equation.** Any such equation can be solved by the method of completing the square. This method gets its name from the fact that in the course of the solution there is added to each member of the equation a number making one member a perfect square.

ORAL EXERCISES

What terms should be added in order to make the following expressions perfect squares?

1. $x^2 + 2x + ?$	**5.** $x^2 + x + ?$	**9.** $x^2 + \frac{3}{5}x + ?$
2. $x^2 - 2x + ?$	**6.** $x^2 - 3x + ?$	**10.** $x^2 - \frac{1}{2}x + ?$
3. $x^2 - 6x + ?$	**7.** $x^2 + \frac{2}{3}x + ?$	**11.** $x^2 + \frac{m}{3}x + ?$
4. $x^2 + 8x + ?$	**8.** $x^2 - \frac{4}{5}x + ?$	**12.** $x^2 + ax + ?$

EXAMPLE

Solve $5x^2 - 3x - 2 = 0$ and check the result.

Solution. $$5x^2 - 3x - 2 = 0. \quad (1)$$

Transposing, $$5x^2 - 3x = 2.$$

Dividing by the coefficient of x^2, $$x^2 - \frac{3}{5}x = \frac{2}{5}.$$

Adding $(-\frac{3}{10})^2$ to each member,

$$x^2 - \frac{3}{5}x + \frac{9}{100} = \frac{2}{5} + \frac{9}{100} = \frac{49}{100},$$

or $$(x - \frac{3}{10})^2 = \frac{49}{100}.$$

Extracting the square root of each member,

$$x - \tfrac{3}{10} = \pm \tfrac{7}{10}.$$

Whence $$x = \tfrac{3}{10} \pm \tfrac{7}{10} = 1 \text{ and } -\tfrac{2}{5}.$$

Check. Substituting 1 for x in (1),

$$5 \cdot 1^2 - 3 \cdot 1 - 2 = 0,$$
$$5 - 3 - 2 = 0,$$
$$0 = 0.$$

Substituting $-\tfrac{2}{5}$ for x in (1),

$$5\left(-\tfrac{2}{5}\right)^2 - 3\left(-\tfrac{2}{5}\right) - 2 = 0,$$
$$\tfrac{4}{5} + \tfrac{6}{5} - 2 = 0,$$
$$0 = 0.$$

The method of solving a quadratic equation illustrated in the preceding example is stated in the

Rule. *Transpose so that the terms containing **x** are in the first member and those which do not contain **x** are in the second.*

Divide each member of the equation by the coefficient of x^2 *unless that coefficient is* $+1$.

*In the equation just obtained, add to each member the square of one half the coefficient of **x**, thus making the first member a perfect trinomial square.*

Rewrite the equation, expressing the first member as the square of a binomial and the second member in its simplest form.

Extract the square root of both members of the equation, and write the sign $\pm$ *before the square root of the second member, thus obtaining two linear equations.*

Solve the equation in which the second member is taken with the sign $+$, *and then solve the equation in which the second member is taken with the sign* $-$. *The results are the roots of the quadratic.*

Check. *Substitute each result separately in place of x in the original equation. If the resulting equations are not obvious identities, simplify each until it becomes one.*

EXERCISES

Solve by completing the square and check real results as directed by the teacher:

1. $x^2 + 4x + 3 = 0.$

2. $x^2 - 2x - 8 = 0.$

3. $s^2 - s - 2 = 0.$

4. $x^2 + 2 = -3x.$

5. $2x^2 + 5x + 3 = 0.$

6. $3x^2 + 7x - 6 = 0.$

7. $2x^2 - 3x - 5 = 0.$

8. $5x^2 - 7x - 6 = 0.$

9. $6x^2 + x - 35 = 0.$

10. $12x^2 - 25x + 12 = 0.$

11. $3q^2 + 8q + 4 = 0.$

12. $x + 2 = 6x^2.$

13. $x - 4 + x^2 = 6 - 2x^2 + 8x$

14. $x^2 = \dfrac{x + 14}{4}.$

15. $(3x - 2)^2 + (x - 1)^2 = 1.$

In Exercises 16–25 obtain results to three decimal places:

16. $x^2 - 12x + 31 = 0.$

HINTS. By applying the rule we get

$$x = 6 + \sqrt{5},$$

and

$$x = 6 - \sqrt{5}.$$

From the table on page **274**, $\sqrt{5} = 2.236.$

Hence we get the result, to three decimals,

$$x = 8.236 \text{ and } 3.764.$$

17. $x^2 - x - 2 = 0.$

18. $n^2 + 2 = 5n.$

19. $3x^2 - 12x + 9 = 2.$

20. $5x^2 + 8x + 2 = 0.$

21. $10 - 2x^2 = x^2 - 7x.$

22. $3x^2 - 18 = 6x - \sqrt{2}.$

23. $x^2 + 2x\sqrt{2} - 6 = 0.$

24. $\dfrac{3}{2}x - \dfrac{1}{4} = \dfrac{x^2}{3}.$

25. $(y+1)(y+2) = 3y(y-4).$

26. $x^4 - 3x^2 + 2 = 0.$

NOTE. This is not a quadratic equation, but many equations of this form can be solved by the methods applicable to quadratics.

Solution.
$$x^4 - 3x^2 + 2 = 0.$$
$$x^4 - 3x^2 = -2.$$
$$x^4 - 3x^2 + \tfrac{9}{4} = -2 + \tfrac{9}{4} = \tfrac{1}{4}.$$
$$(x^2 - \tfrac{3}{2})^2 = \tfrac{1}{4}.$$
$$x^2 - \tfrac{3}{2} = \pm \tfrac{1}{2}.$$
$$x^2 = 2 \text{ or } 1.$$

Whence
$$x = \pm\sqrt{2}, \pm 1.$$

Check as usual.

NOTE. It should be particularly observed that the equation of Exercise 26 has four roots instead of two. In general an equation has a number of roots equal to its degree. Thus the equations in Exercises 29 and 31 have six and eight roots respectively, although some of them are imaginary and the student at present should not be required to find them at all.

27. $x^4 - 5x^2 + 4 = 0.$

28. $x^4 - 13x^2 + 36 = 0.$

29. $x^6 + 8 = 9x^3.$

30. $x^6 - 7x^3 = 8.$

31. $x^8 - 17x^4 + 16 = 0.$

32. $6x^4 - 11x^2 + 3 = 0.$

33. $4m^6 - 15 = 7m^3.$

34. $x^4 - 4x^2 + 3 = 0.$

35. $y^6 + 7y^3 = 8.$

36. $m^4 - 4\sqrt{2}\,m^2 + 7 = 0.$

37. $(x^2 - 2x)^2 - 7(x^2 - 2x) = -12.$

Solution. Let
$$x^2 - 2x = y.$$

Substituting y for $x^2 - 2x$, we obtain
$$y^2 - 7y = -12.$$

Solving,
$$y = 3 \text{ and } 4.$$

Then
$$x^2 - 2x = 3.$$

Whence
$$x = 3 \text{ and } -1.$$

Also
$$x^2 - 2x = 4.$$

Whence
$$x = 1 \pm \sqrt{5}.$$

In Exercises 38–41 do not expand, but solve as in Exercise 37:

38. $(x-1)^2+4(x-1)=5.$

39. $(x^2-4x)^2-5(x^2-4x)-24=0.$

40. $3(y^2+3y)^2-7(y^2+3y)=20.$

41. $\left(x-\frac{6}{x}\right)^2+4\left(x-\frac{6}{x}\right)-5=0.$

42. $ax^2+bx+c=0.$

Solution. $ax^2+bx+c=0.$

$$x^2+\frac{b}{a}x=-\frac{c}{a}.$$

$$x^2+\frac{b}{a}x+\left(\frac{b}{2a}\right)^2=\frac{b^2}{4a^2}-\frac{c}{a}=\frac{b^2-4ac}{4a^2}.$$

$$x+\frac{b}{2a}=\pm\frac{\sqrt{b^2-4ac}}{2a}.$$

$$x=-\frac{b}{2a}\pm\frac{\sqrt{b^2-4ac}}{2a}=\frac{-b\pm\sqrt{b^2-4ac}}{2a}.$$

43. $x^2+3ax+2a^2=0.$

44. $2y^2+by=6b^2.$

45. $bx^2+x=1+bx.$

46. $bx^2-a(b+1)x+a^2=0.$

47. $\frac{b}{2a}x^2=\frac{b}{2a}+1-x.$

48. $cx^2-(c^2+d)x+cd=0.$

49. $abx^2-x(a+b)\sqrt{ab}+ab=0.$

50. $3x^4-a^2x^2-2a^4=0.$

51. $6a^2y^2-7ary+2r^2=0.$

52. $(3ax+4b)^2=(2ax-b)^2.$

53. $\frac{3}{4}x^2-2ax+\frac{1}{2}a^2=\frac{9a^2}{4}.$

54. $a^2x^2+3ax\sqrt{b}-10b=0.$

55. $9k^2-x^2=2kx\sqrt{3}.$

56. $(ax+b)^2+3(ax+b)+2=0.$

57. $(a^2x^2-3ax)^2=14(a^2x^2-3ax)-40.$

88. Solution by formula. In Exercise 42, above, the general quadratic $ax^2 + bx + c = 0$ has been solved and the roots found to be

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}. \qquad (F)$$

The expression (F) is a general result and may be used as a formula to solve any quadratic equation in the standard form $ax^2 + bx + c = 0$, in which a, b, and c may represent numbers, single letters, binomials, or any other form of algebraic expression not involving x.

If the numbers a, b, and c are such that the expression $b^2 - 4ac$ is negative, the formula contains the square root of a negative number, which is a kind of number not yet fully considered in this text. In the exercises that follow it will be assumed that only such numerical values of the literal coefficients are involved as will not make $b^2 - 4ac$ negative. A discussion of the case here ruled out will be found in Chapter XII.

EXERCISES

Solve for x by formula and check the results as directed by the teacher:

1. $4x^2 + 8x = 3$.

Solution. Writing in standard form,

$$4x^2 + 8x - 3 = 0.$$

Comparing with $ax^2 + bx + c = 0$, evidently 4 corresponds to a, 8 to b, and -3 to c. Substituting the values in (F) gives

$$x = \frac{-8 \pm \sqrt{64 - 4 \cdot 4 \cdot (-3)}}{2 \cdot 4}$$

$$= \frac{-8 \pm \sqrt{64 + 48}}{8} = \frac{-8 \pm \sqrt{112}}{8}$$

$$= \frac{-8 \pm 4\sqrt{7}}{8} = -1 \pm \tfrac{1}{2}\sqrt{7}.$$

Check as usual.

2. $x^2 - 6x - 16 = 0.$

3. $x^2 - x - 1 = 0.$

4. $3x^2 + 7x + 3 = 0.$

5. $4x^2 + 5x = 3.$

6. $2x + 4 = x^2.$

7. $x = 1 - x^2.$

8. $4x^2 - 11x - 60 = 0.$

9. $3x + 4 = x^2.$

10. $11x^2 - 15x = 26.$

11. $\frac{3}{2}x - \frac{1}{2} = x^2.$

12. $x + \frac{3}{2} = \frac{2}{3}x^2.$

13. $\frac{3}{5} - \frac{1}{3}x^2 = 2x.$

14. $6x = 1 + 2x^2.$

15. $x^2 + 2x = 8 - x^2.$

16. $3p^2x^2 = px + 4.$

Solution. Writing in standard form,

$$3p^2x^2 - px - 4 = 0.$$

Here $a = 3p^2,\quad b = -p,\quad$ and $\quad c = -4.$

Substituting these values in the formula (F),

$$x = \frac{-(-p) \pm \sqrt{(-p)^2 - 4 \cdot 3p^2(-4)}}{2 \cdot 3p^2}$$

$$= \frac{p \pm \sqrt{p^2 + 48p^2}}{6p^2} = \frac{p \pm 7p}{6p^2} = \frac{4}{3p} \text{ and } \frac{-1}{p}.$$

Check as usual.

17. $x^2 - 2kx - 3k^2 = 0.$

18. $3a^2 = 7ax + 6x^2.$

19. $4x^2 + kx - 14k^2 = 0.$

20. $bx = 12x^2 - b^2.$

21. $a^2x^2 + 4abx + 3b^2 = 0.$

22. $12p^2x^2 - 4prx - r^2 = 0.$

23. $3a^2x^2 + 8abx + 4b^2 = 0.$

24. $x - 3\sqrt{c} - \frac{4c}{x} = 0.$

25. $6h^2 + \frac{13hn}{x} = \frac{8n^2}{x^2}.$

26. $x^2 + bx + \frac{1}{c} = 0.$

27. $\frac{x^2}{a} + x + 3b = 0.$

28. $\frac{bx^2}{a} - \frac{x}{a} - \frac{1}{a} = 0.$

29. $13a^2b^2x^2 = 9b^4 + 4a^4x^4.$

30. $2x^2 + 5x = cx^2 + 3cx + 3.$

Solution. $(2-c)x^2 + (5-3c)x - 3 = 0.$

Here $a = 2 - c, \quad b = 5 - 3c, \quad$ and $\quad c = -3.$

Hence $$x = \frac{3c - 5 \pm \sqrt{(5-3c)^2 + 12(2-c)}}{2(2-c)}$$

$$= \frac{3c - 5 \pm \sqrt{49 - 42c + 9c^2}}{4 - 2c}.$$

$$= \frac{3c - 5 \pm (7 - 3c)}{4 - 2c}$$

$$= \frac{2}{4-2c}, \text{ or } \frac{1}{2-c}, \text{ and } \frac{6c-12}{4-2c}, \text{ or } -3.$$

Check as usual.

31. $p^2 + x^2 = 2px + 2x - 2p.$

32. $(x-1)^2 = a(x - x^2).$

33. $x^2 + 2x = kx^2 + kx - 1.$

34. $3x + ax^2 = 2(x^2 + ax - 1).$

35. $x^2 - cx = \frac{1}{2}(ax - ac).$

36. $4a^2x^4 + 4c^2 = 16a^2x^2 + c^2x^2.$

37. $x^6 - a^3x^3 = 8b^3x^3 - 8a^3b^3.$

38. $a^2 + x^2 = 2ax + b^2.$

39. $12pq = x^2 + 4qx - 3px.$

40. $x^4 - a^2x^2 + a^2b^4 = b^4x^2.$

89. Comparison of the various methods. Four methods have been given for the solution of the quadratic equation:

(*a*) Solution by factoring.
(*b*) Solution by graphing.
(*c*) Solution by completing the square.
(*d*) Solution by formula.

In practice, the first and the last of these methods are most convenient.

When an equation with integral coefficients can be solved by factoring, as explained in Chapter III, the roots are always rational numbers.

For example, $3x^2 + 2x - 8 = 0$ factors into $(3x - 4)(x + 2) = 0$. Hence the roots are $x = \frac{4}{3}$ and -2.

An inspection of the formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

shows that, if a, b, and c are integers, one always obtains roots involving radicals unless the expression under the radical sign, $b^2 - 4ac$, is a perfect square. In this case, however, the values of the roots are rational numbers.

For example, in the equation $3x^2 + 2x - 8 = 0$ the value of the expression $b^2 - 4ac$ is $(2)^2 - 4 \cdot 3(-8) = 100$, which is a perfect square. Hence the roots of $3x^2 + 2x - 8 = 0$ are rational numbers, since the radical term in the roots can be expressed as a rational number.

Hence to determine whether a quadratic equation of the form $ax^2 + bx + c = 0$ can be solved by factoring into rational factors, we have the

Rule. *Compute the value of $b^2 - 4ac$ for the equation.*

If the result is the square of an integer, the left member of the equation can be factored and the roots are rational.

ORAL EXERCISES

Determine which of the following can be solved by factoring:

1. $x^2 + 10x + 25 = 0$.	**6.** $x^2 - 4x - 4 = 0$.
2. $x^2 - 5x - 400 = 0$.	**7.** $3x^2 - x + 4 = 0$.
3. $x^2 - 7x - 3 = 0$.	**8.** $3x^2 - 5x + 8 = 0$.
4. $2x^2 + 3x + 1 = 0$.	**9.** $2x^2 - 10x - 25 = 0$.
5. $2x^2 + 3x + 2 = 0$.	**10.** $3x^2 + 2ax - a^2 = 0$.

REVIEW EXERCISES

Solve the following by the method best adapted to each:

1. $3x^2 - 7x - 10 = 0.$
2. $5x^2 + 14x + 8 = 0.$
3. $4x^2 + 3x - 2 = 0.$
4. $x^2 - 1.7x - .84 = 0.$
5. $8x^2 + 2\sqrt{5}x - 15 = 0.$
6. $x^4 - 7x^2 + 12 = 0.$
7. $6x^6 - 20 = 2x^3.$
8. $x^3 - x = 0.$
9. $x^4 = x.$
10. $x^4 + 64x = 0.$
11. $(x^2 - 3x)^2 - 2(x^2 - 3x) = 8.$
12. $5x^2 - 9x + 3 = 0.$
13. $.03x^2 + .01x = .1.$
14. $x^2 + x\sqrt{2} - \sqrt{6} = x\sqrt{3}.$
15. $\dfrac{x-1}{x} = \dfrac{x+1}{6}.$
16. $\dfrac{x-3}{x+4} = \dfrac{x-4}{2x+6}.$
17. $\dfrac{x+1}{x+2} + \dfrac{x+3}{x+4} = \dfrac{22}{15}.$
18. $\dfrac{x^2-1}{x^2+1} = \dfrac{x^2+1}{x^2} - 2.$
19. $\dfrac{14}{x+4} + \dfrac{x-1}{x-2} = \dfrac{x+1}{x+2}.$
20. $\dfrac{2x-1}{x-1} + \dfrac{1-x}{x-2} = \dfrac{x+1}{x+2}.$
21. $pqx^2 - rqx + psx = rs.$
22. $x^2 - 2ax + a^2 - b^2 = 0.$
23. $x^2 - 2ax + a^2 + b^2 = 0.$
24. $(x^2 + 5x + 2)^2 - 6(x^2 + 5x + 2) = 16.$
25. $\dfrac{3x-1}{5x+1} - \dfrac{x+1}{x-1} = \dfrac{48}{(5x+1)(1-x)}.$
26. $x - 3 = \dfrac{x^3 - 10x^2 + 1}{(x-3)^2}.$
27. $15x^2 - 1.95x + .054 = 0.$
28. $10 - 9x = 7x^2.$
29. $x^2 + 961a^2 = 62ax.$
30. $(x^3 + 4)^2 = 4 + x^3.$
31. $(2x + 3)(x - 4) = (3x - 8)(4x - 1).$
32. $(3x - 2)(x - 5) = (4x - 3)(x + 1).$

33. $\dfrac{1}{x+7}=\dfrac{2}{5}-\dfrac{1}{3-x}$.

34. $\dfrac{x}{x^2+3x-4}=\dfrac{x+2}{x+4}$.

35. $\dfrac{x+3}{x^2-3x+2}=\dfrac{x-1}{x-2}-\dfrac{x-2}{x-1}$.

36. $\dfrac{5x-11}{x-1}+\dfrac{1-3x}{x+2}=\dfrac{2x-1}{x+1}$.

37. $\dfrac{x+12}{x+6}-7=\dfrac{x+7}{x+5}$.

38. $17x=6x^2-10$.

39. $7x^4-42x^2+25=13x^2-5x^4+400$.

40. $2k^2x^2+8kx=5(3kx-1)$.

41. $37pqx=210q^2-p^2x^2$.

42. $\dfrac{x+1}{x^3-1}-\dfrac{x-1}{x^3+1}=\dfrac{2x+1}{(x^2+x+1)(x^2-x+1)}$.

43. $\dfrac{3x-2}{x-1}+\dfrac{2}{3}=9\tfrac{1}{6}-\dfrac{5(x+1)}{4x-1}$.

44. $x^4-x^2=4(9+x^2)$.

45. $3+\dfrac{4k}{c-2x}=\dfrac{k-2x}{c}$.

46. $\dfrac{4-3x}{x}-\dfrac{2x}{1-x}=\dfrac{2x+5}{3-2x}$.

47. $\dfrac{6x^2+x-5}{4x^2-x-2}=\dfrac{12x^2+2x-4}{8x^2-2x+4}$.

48. $8.4x^2+.005x-.15=0$.

49. $324x^4+1936=1665x^2$.

50. $3x^2+.7x=.2$.

51. $3a^2x^2-10abx-25b^2=2(2abx-a^2x^2)$.

52. $\dfrac{1}{x^2-3x+2}+\dfrac{1}{x^2+x-6}=\dfrac{1}{3-2x-x^2}.$

53. $\dfrac{x+2}{(x+4)(x-1)}-\dfrac{2}{(x+4)(3-x)}=\dfrac{3(2-x)}{(x-3)(1-x)}.$

54. $\dfrac{2(x^2-2)-1}{x^2-1}=\dfrac{x^2+2}{(x^2-2)-4}.$

55. $(3x-7)(2x+1)-(5x+2)(2x-3)=0.$

56. $(x-2)^2-(1-2x)(3x+5)=5-(1-2x)(3x+2).$

57. $(x^2-2x-2)^2+2(x^2-2x-2)-3=0.$

58. $(x^2-4x+1)^2-4(x^2-4x)-16=0.$

59. $(2x-b)^2=a(2x-b)+2a^2.$

60. $x+\dfrac{1}{x}=m+\dfrac{1}{m}.$

PROBLEMS

1. Separate 42 into two parts such that the first shall be the square of the second.

2. Find two consecutive odd numbers whose product is 143.

3. The sum of the reciprocals of two consecutive numbers is $\frac{13}{42}$. Find the numbers.

4. The time, in hours, required for a trip of 216 miles was 6 greater than the rate, in miles per hour. Find the time and the rate.

5. The altitude of a triangle is 6 feet less than the base. The area is 56 square feet. Find the base and altitude.

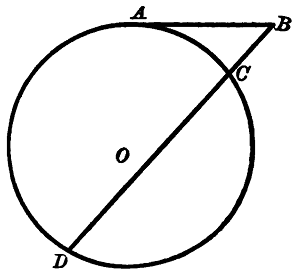
6. One leg of a right triangle is 9 feet shorter than the other, and the area is 45 square feet. Find the three sides.

7. The area of a trapezoid is 180 square feet. One base is 4 feet greater than the altitude, the other is three times the altitude. Find the two bases and the altitude.

8. A rectangle whose area is 27 square inches has a perimeter of 21 inches. Find the length of the shorter side.

9. A polygon of n sides always has $\frac{1}{2}n(n-3)$ diagonals. How many sides has a polygon with 119 diagonals?

10. If AB, in the accompanying figure, is a tangent to the circle and BD is any secant, then $(AB)^2 = BC \cdot BD$. If $AB = 6$ and $CD = 10$, find BC.



11. A requires 4 more days than B to do a piece of work. Working together they require $2\frac{2}{3}$ days. How many days will each require alone?

12. In selling an article at \$20, a merchant's per cent of profit is 9 greater than the original cost of the article in dollars. Find the original cost and the per cent of profit.

13. A 3-inch square is cut from each corner of a square piece of tin. The sides are then turned up to form an open box of volume 12 cubic inches. What was the side of the original square?

14. The number of lines on a certain printed page is 20 less than the average number of letters per line. If the number of letters per line is decreased by 16, the number of lines must be increased by 15 in order that the new page may contain as much matter as the old. How many lines and how many letters were there on the original page?

15. For what positive value or values of x will the product of $x-5$ and $x+5$ be 1 greater than their difference?

16. What values of x will make the product of $4x-5$ and $x-2$ equal in value to $x+4$?

17. The length of a rectangular room exceeds its width by 2 feet. A rug placed in the middle of the floor leaves a margin of 1 foot all around. If the area of the rug is eight times that of the margin, find the dimensions of the room.

18. A sum of \$4000 is invested, and at the end of each year the year's interest, plus \$400, is added to the investment. At the beginning of the third year the investment amounts to \$5230. What is the rate of interest?

19. The cost of an outing was \$60. If there had been two more in the party, the share of each would have been a dollar less. How many were there?

20. The dimensions of a rectangular box are expressed by three consecutive numbers. Its surface is 214 square inches. Find its dimensions.

21. The radius of a circle is 28 inches. How much must this be shortened in order to decrease the area of the circle by 1078 square inches?

22. Two bodies, A and B, move on the sides of a right triangle. A is now 5 feet from the vertex, and moving from it at the rate of 10 feet per second. B is 35 feet from the vertex, and moving toward it at 5 feet per second. At what time (past or future) are they 75 feet apart?

23. How high is a mountain which can just be seen from a point on the surface of the sea, 40 miles distant?

HINT. See Exercise 10.

24. The distance in feet, s, through which a body falls from rest in t seconds, is given by the formula $s = \frac{1}{2}gt^2$, where $g = 32$, approximately. A bomb dropped from an aëroplane struck the ground below 8 seconds later. How high was the aëroplane at the time?

25. If a stone be thrown vertically upward, with a velocity of 100 feet per second, it is known that, neglecting the resistance of the air, its height after t seconds will be $100t - 16t^2$ feet. After how many seconds will it be 136 feet high? Explain the double answer.

26. After how many seconds will the stone of Exercise 25 return to its starting point? Explain the zero root.

CHAPTER XI

IRRATIONAL EQUATIONS

90. Definitions and discussion. An **irrational** equation in one unknown is an equation in which the unknown occurs under a radical, or is affected by a fractional exponent.

Thus $3x + 5\sqrt{x} = 1$, $x - x^{\frac{1}{2}} + 1 = 0$, and $\sqrt[3]{x^2 - 3x + 2} = 6$ are irrational equations.

The chief difficulty involved in the solution of such equations arises from the fact that sometimes results are obtained which do not satisfy the given equation and hence are not roots of that equation. A result of this kind is called **extraneous.**

EXAMPLE

(*a*) Solve $\sqrt{x - 6} - 4 = 0$.

Solution. Transposing,

$$\sqrt{x-6} = 4. \quad (1)$$

Squaring, $$x - 6 = 16. \quad (2)$$

Solving, $$x = 22.$$

Check. $$\sqrt{22 - 6} - 4 = 0.$$

$$\sqrt{16} - 4 = 0.$$

$$4 - 4 = 0,$$

which is true.

(*b*) Solve $-\sqrt{x - 6} - 4 = 0$.

$$-\sqrt{x-6} = 4. \quad (1)$$

$$x - 6 = 16. \quad (2)$$

$$x = 22.$$

$$-\sqrt{22 - 6} - 4 = 0.$$

$$-\sqrt{16} - 4 = 0.$$

$$-4 - 4 = 0,$$

which is not true.

It appears from a study of these solutions that statements (1) differ only in the signs preceding their left members. Consequently this distinction disappears after

squaring, and equations (2) are identical. Since the remainder of the work in both (*a*) and (*b*) consists in the solution of (2), the result obtained is really the root of this equation. Whether the root obtained satisfies both (*a*) and (*b*), or only one of them, can be determined only by substitution. Hence it appears that (*a*) is an equation and that (*b*) is not, but is merely a false statement in the form of an equation.

In any case, all of the roots of the original equation are sure to be among the results found, provided no factor containing the unknown has been divided out. But no result should be called a root unless it satisfies the original equation. This means that all results must be checked.

In irrational equations, as in all the work up to the present, it is understood that unless a radical or an expression affected by a fractional exponent is preceded by the double sign $\pm$ it has only the one value, just like any other number symbol.

Thus $\sqrt{16}$ means $+4$, and not -4.
Also $4^{\frac{1}{2}}$ means $+2$, while $-4^{\frac{1}{2}}$ means $-\sqrt{4}$, or -2.

If this fact is kept in mind, it is clear from an inspection of (*b*), above, that it could have no root, since the sum of two negative numbers could not possibly be zero.

The method of solving equations containing radicals is stated in the

Rule. *Transpose the terms so that one radical expression (the least simple one if there are two or more) is the only term in one member of the equation.*

Next raise both members of the resulting equation to the same power as the index of this radical.

Combine like terms in each member and, if radical expressions still remain, repeat the two preceding operations

obtained which is free from radicals;
ı.

he original equation the values found
numerical equation to its simplest
but not by raising both members of

ous roots.

RCISES

oots and check results:

7. $\sqrt{x+1}=\sqrt{3x-5}$.

8. $2\sqrt{8x}=x\sqrt{4}$.

. $3\sqrt{2x+6}=\sqrt{6x^2-6}$.

$\sqrt[3]{4x+3}-\sqrt[3]{4-3x}=0$.

$\sqrt[3]{2x+2}=\sqrt{2x-2}$.

$\overline{x+1}=\sqrt{x-1}$.

-1.

$\sqrt{x}+1$.

$\therefore$

$=\frac{9}{4}$.

ation,

$\overline{3x-1}$.

20. $x + 4 + \sqrt{x} = \sqrt{x^2 + 16}$.

21. $\sqrt{2x - 4} + \sqrt{x + 6} = \sqrt{7x - 6}$.

22. $\sqrt{x + 2} + \sqrt{x - 1} - \sqrt{3x + 3} = 0$.

23. $\sqrt{3x + 1} - \sqrt{x + 1} = \sqrt{x - 4}$.

24. $x^{\frac{4}{3}} - 9x^{\frac{2}{3}} + 20 = 0$.

NOTE. The equation here given is not a quadratic equation, but it is of the general type $ax^{2n} + bx^n + c = 0$. Here x occurs in but two terms, and its exponent in one term is twice that in the other term. Many equations in this form can be solved by completing the square (compare Exercise 42, p. 143).

Solution.

$$x^{\frac{4}{3}} - 9x^{\frac{2}{3}} + \tfrac{81}{4} = -20 + \tfrac{81}{4}.$$

$$x^{\frac{2}{3}} - \tfrac{9}{2} = \pm \tfrac{1}{2}.$$

$$x^{\frac{2}{3}} = 5 \text{ or } 4.$$

Whence $$x = \pm 5^{\frac{3}{2}} \text{ or } \pm 8.$$

Check. Substituting $\pm 5^{\frac{3}{2}}$ for x in the original equation,

$$\left(\pm 5^{\frac{3}{2}}\right)^{\frac{4}{3}} - 9\left(\pm 5^{\frac{3}{2}}\right)^{\frac{2}{3}} + 20 = 0,$$

or $$5^2 - 9 \cdot 5 + 20 = 0, \text{ or } 0 = 0.$$

Substituting ± 8 for x in the original equation,

$$(\pm 8)^{\frac{4}{3}} - 9(\pm 8)^{\frac{2}{3}} + 20 = 0.$$

$$16 - 36 + 20 = 0, \text{ or } 0 = 0.$$

25. $x^3 + 5x^{\frac{3}{2}} - 14 = 0$.

26. $x^3 - 10x^{\frac{3}{2}} = 11$.

27. $6 = x^{\frac{1}{2}} + x$.

28. $3x^{\frac{4}{3}} + 5x^{\frac{2}{3}} + 2 = 0$.

29. $x^{-4} - 17x^{-2} + 52 = 0$.

30. $2x^{\frac{2}{3}} - 7\sqrt[3]{x} + 6 = 0$.

31. $x^{-\frac{3}{2}} = 4x^{-\frac{3}{4}} + 32$.

32. $x^{\frac{1}{3}} - x^{\frac{1}{6}} - 6 = 0$.

33. $x^{-1} - \dfrac{25}{36x^{\frac{1}{2}}} + \dfrac{1}{9} = 0$.

34. $7x^{\frac{3}{5}} - 6 = 2x^{\frac{6}{5}}$.

35. $x^2 + 5x + 3\sqrt{x^2 + 5x} - 54 = 0$.

HINT. Let $y = x^2 + 5x$.

36. $3x^2 - 4x - 11\sqrt{3x^2 - 4x} + 28 = 0.$

37. $2x^2 - 3x - 4 - \sqrt{2x^2 - 3x - 1} + 1 = 0.$

38. $x^2 - 2x - 5\sqrt{x^2 - 2x - 4} + 2 = 0.$

39. $\sqrt{2x - 4} + 5 = 1.$

40. $\sqrt[3]{x + 3} = \sqrt{\frac{1}{2}x - 3}.$

41. $12x^{\frac{4}{3}} - 27x^{\frac{2}{3}} = 20x^{\frac{2}{3}} - 45.$

42. $15 - 2\sqrt{x^2 + 9} = x^2 + 9.$

43. $27\left(8 + 27x^{\frac{3}{2}} - x^3\right) = 8x^{\frac{3}{2}}.$

44. $4\sqrt{x + 2} = 3\sqrt{x + 4}.$

45. $\sqrt[4]{x + 4} = \sqrt{x - 8}.$

46. $\dfrac{6\sqrt{x} - 8}{\sqrt{2x} - \sqrt{8}} = \sqrt{2}.$

47. $\dfrac{2\sqrt{3}}{\sqrt{2x - 3}} = \dfrac{\sqrt{2x + 12}}{3\sqrt{3}}$

48. $\sqrt{x + 1} = \sqrt{x - 4} + 1.$

49. $4(x - 5) - 2(x - 5)^{\frac{1}{2}} - 2 = 0.$

50. $4x\sqrt{x} = (x + 3)\sqrt{9x}.$

51. $\dfrac{\sqrt{x + 2}}{\sqrt{8 - x}} - \dfrac{\sqrt{8 - x}}{\sqrt{x + 2}} = \dfrac{8}{3}.$

52. $5x^{\frac{3}{2}} + 37x^{\frac{3}{4}} - 200 = 16 - 3x^{\frac{3}{2}}.$

53. $4\sqrt{x} - 21\sqrt[4]{x} + 27 = 0.$

54. $(x^2 - 5x + 2)^{\frac{1}{2}} - 5(x^2 - 5x + 2)^{\frac{1}{4}} + 6 = 0.$

55. $\sqrt{17 + 2\sqrt{3 + s + \sqrt{s + 7}}} - 5 = 0.$

56. $3x^2 - x - 22 = 6\sqrt{3x^2 - x - 6}.$

57. If $t = \pi\sqrt{\dfrac{l}{g}}$, solve for l and g.

58. If $r = \frac{1}{2}x\sqrt{3}$ and $A = 2\pi r^2$, express A in terms of x.

59. If $a^2 = 2r^2$ and $C = 2\pi r$, express C in terms of a.

60. If $A = \dfrac{3r^2}{2}\sqrt{3}$ and $x = \frac{1}{2}r\sqrt{3}$, express A in terms of x.

61. If $K = 3r^2$ and $a = \frac{1}{2}r\sqrt{3 + \sqrt{2}}$, express K in terms of a.

CHAPTER XII

IMAGINARIES

91. Definitions. When the square root of a negative number arose in our previous work, it was called an imaginary, but no attempt was then made to use it or to explain its meaning. The treatment of imaginaries was deferred because there were so many topics of more importance to the beginner. It must not be supposed, however, that imaginaries are not of great value in mathematics. They are frequently used in certain branches of applied science; and it is unfortunate that symbols which can be employed in numerical computations to obtain practical results should ever have been called imaginary. By such a name something unreal and fanciful is suggested, to obviate which it has been proposed to call imaginary numbers *orthotomic numbers*, but this name has been little used.

The equation $x^2+1=0$, or $x^2=-1$, states that x is a number whose square is -1. By defining a new number, $\sqrt{-1}$, as one whose square is -1, we obtain one root for the equation $x^2+1=0$.

Similarly, $\sqrt{-5}$ is a number whose square is -5. And, in general, $\sqrt{-n}$ is a number whose square is $-n$. Obviously, $\sqrt{-5}$ means something very different from $\sqrt{5}$, and $\sqrt{-n}$ from $\sqrt{n}$.

The positive numbers are all multiples of the unit $+1$, and the negative numbers are all multiples of the unit -1. Similarly, pure imaginary numbers are real multiples of the imaginary unit $\sqrt{-1}$, as $2\sqrt{-1}$, $-5\sqrt{-1}$, and $b\sqrt{-1}$.

Furthermore $\sqrt{-4} = 2\sqrt{-1}$; $\sqrt{-a^2} = a\sqrt{-1}$; and $\sqrt{-5} = \sqrt{5} \cdot \sqrt{-1}$.

The imaginary unit $\sqrt{-1}$ is often denoted by the letter i; that is, $3\sqrt{-1} = 3i$.

If a real number be united to a pure imaginary by a plus sign or a minus sign, the expression thus obtained is called a **complex number**.

Thus $-2 + \sqrt{-1}$ and $3 - 2\sqrt{-4}$ are complex numbers. The general form of a complex number is $a + bi$, in which a and b may be any real numbers.

NOTE. Up to the time of Gauss (1777–1855) complex numbers were not clearly understood and were usually thought of as absurd. The situation reminds one of the time when negative numbers were similarly regarded, and the veil was removed from both in about the same way. It was found that negative numbers really had a significance — that they could be used in problems that involve debt, opposite directions, and many other everyday relations. The interpretation of imaginary numbers is not quite so obvious, and is not considered in this text. But as soon as it was seen that an interpretation was possible the ice was broken, and it needed only the insight and authority of a man like Gauss to give complex numbers their proper place in mathematics.

ORAL EXERCISES

Express as multiples of $\sqrt{-1}$, or i:

1. $\sqrt{-16}$.	5. $3\sqrt{-6}$.	9. $\sqrt{5} \cdot \sqrt{-20}$.
2. $\sqrt{-25}$.	6. $2\sqrt{-c}$.	10. $a\sqrt{-c}$.
3. $\sqrt{-a^2}$.	7. $\sqrt{3} \cdot \sqrt{-3}$.	11. $\sqrt{-x^2-2x-1}$.
4. $2\sqrt{-3}$.	8. $\sqrt{2} \cdot \sqrt{-5}$.	12. $\sqrt{-y^2+6y-9}$.

92. Addition and subtraction of imaginaries. The fundamental operations of addition and subtraction are performed on imaginary and complex numbers as they are performed on rational numbers and ordinary radicals of the same form.

Thus $$2\sqrt{-1}+4\sqrt{-1}=6\sqrt{-1},$$

and $$5\sqrt{-1}-3\sqrt{-1}=2\sqrt{-1}.$$

Also $(3+5\sqrt{-1})+(4-2\sqrt{-1})=7+3\sqrt{-1}.$

Similarly, $$(a+bi)+(c+di)=a+c+(b+d)i.$$

EXERCISES

Simplify:

1. $3\sqrt{-1}+2\sqrt{-1}.$
2. $4\sqrt{-1}+\sqrt{-4}.$
3. $\sqrt{-25}-\sqrt{-16}.$
4. $\sqrt{-9}+\sqrt{-4}.$
5. $\sqrt{-4}+\sqrt{-16}.$
6. $(-8)^{\frac{1}{2}}+(-32)^{\frac{1}{2}}.$
7. $\sqrt{-18}+\sqrt{-8}.$
8. $4\sqrt{-25x^2}-2\sqrt{-36x^2}.$
9. $2+3\sqrt{-1}+6-5\sqrt{-1}.$
10. $7\sqrt{-a^2}-5a+4\sqrt{-a^2}.$
11. $(3a-6ib)+(a+ib).$
12. $4-8i+16-3\sqrt{-9}.$
13. $5+3\sqrt{-49x^2}-6\sqrt{-16x^2}+4.$
14. $18-3(-1)^{\frac{1}{2}}+6(-25)^{\frac{1}{2}}+4.$
15. $5\sqrt{-3}+3\sqrt{-2}-\sqrt{-27}+2\sqrt{-8}.$
16. $(8-5\sqrt{-16})-(7+3\sqrt{-25}).$
17. $4\sqrt{-9a^4}-6a^2\sqrt{-16}+3\sqrt{-6}+5\sqrt{-54}.$
18. $(5x-6iy)-(3x+2iy).$

93. Multiplication of imaginaries. By the definition of square root, the square of $(-n)^{\frac{1}{2}}$ is $-n$.

Therefore
$$(\sqrt{-1})^2=-1.$$
$$(\sqrt{-1})^3=(\sqrt{-1})^2\sqrt{-1}=-\sqrt{-1}.$$
$$(\sqrt{-1})^4=(\sqrt{-1})^2(\sqrt{-1})^2=(-1)(-1)=1.$$

To multiply $\sqrt{-2}$ by $\sqrt{-3}$, we first write

$$\sqrt{-2} = \sqrt{2} \cdot \sqrt{-1},$$

and

$$\sqrt{-3} = \sqrt{3} \cdot \sqrt{-1}.$$

Then

$$\sqrt{-2} \cdot \sqrt{-3} = (\sqrt{2} \cdot \sqrt{-1})(\sqrt{3} \cdot \sqrt{-1})$$
$$= \sqrt{6} \cdot \sqrt{-1} \cdot \sqrt{-1} = -\sqrt{6}.$$

Similarly,

$$(2\sqrt{-5})(-3\sqrt{-2}) = (2\sqrt{5} \cdot \sqrt{-1}) \cdot (-3\sqrt{2} \cdot \sqrt{-1})$$
$$= -6\sqrt{10}(-1) = 6\sqrt{10}.$$

In general, if $\sqrt{-a}$ and $\sqrt{-b}$ are two imaginaries whose product is desired, they should first be written in the form $\sqrt{a} \cdot \sqrt{-1}$ and $\sqrt{b} \cdot \sqrt{-1}$ and the multiplication should only then be performed. This method will prevent many errors.

In this connection it must be clearly understood that one rule followed in the multiplication of real radicals (see page 117) does not apply to imaginary numbers.

In the case of ordinary radicals we have

$$\sqrt{2} \cdot \sqrt{3} = \sqrt{2 \cdot 3} = \sqrt{6}.$$

But the product of two imaginaries like $\sqrt{-2} \cdot \sqrt{-3}$ does not equal $\sqrt{(-2)(-3)}$, for this equals $\sqrt{6}$. We have seen above that $\sqrt{-2} \cdot \sqrt{-3} = -\sqrt{6}$.

In multiplying two complex numbers, write each expression in the form $a \pm bi$ and proceed as in the following

EXAMPLE

Multiply $4 + \sqrt{-5}$ by $2 - \sqrt{-6}$.

Solution. $4 + \sqrt{-5} = 4 + \sqrt{5} \cdot \sqrt{-1},$ (1)

$2 - \sqrt{-6} = 2 - \sqrt{6} \cdot \sqrt{-1}.$ (2)

Multiplying (1) by (2), $8 + 2\sqrt{5} \cdot \sqrt{-1} - 4\sqrt{6} \cdot \sqrt{-1} - \sqrt{30}(-1).$

Rewriting, $8 + 2\sqrt{-5} - 4\sqrt{-6} + \sqrt{30}.$

EXERCISES

Perform the following indicated multiplications and simplify results:

1. $(-1)^5$.
2. $(-1)^6$.
3. $(-1)^7$.
4. $(-1)^8$.
5. $(\sqrt{-1})^{-4}$.
6. $(\sqrt{-1})^7$.
7. $(\sqrt{-2})^8$.
8. $(\sqrt{-3})^5$.
9. $(2\sqrt{-5})^7$.
10. $\sqrt{-4}\cdot\sqrt{-25}$.
11. $\sqrt{-4}(-\sqrt{-9})$.
12. $\sqrt{-7}(-\sqrt{-6})$.
13. $\sqrt{-16}\cdot\sqrt{-10}$.
14. $2\sqrt{-3}\cdot 3\sqrt{-2}$.
15. $\sqrt{-a}\cdot\sqrt{-b}$.
16. $3\sqrt{-7}(-2\sqrt{-5})$.
17. $\sqrt{m+n}\cdot\sqrt{-m-n}$.
18. $(3+\sqrt{-1})(3-\sqrt{-1})$.
19. $(5+\sqrt{-3})(5-\sqrt{-3})$.
20. $(3-4\sqrt{2}\,i)(3+2\sqrt{2}\,i)$.
21. $(4+\sqrt{-1})(5-\sqrt{-3})$.
22. $(5-3i)(6-5\sqrt{2}\,i)$.
23. $(a+ib)(c+id)$.
24. $(a+ib)^2$.
25. $(a+ib)(a-ib)$.
26. $(-\frac{1}{2}+\frac{1}{2}\sqrt{-3})^2$.
27. $(-\frac{1}{2}-\frac{1}{2}\sqrt{-3})^2$.
28. $(-2+2\sqrt{-3})^3$.
29. $(-2-2\sqrt{-3})^3$.
30. $(x-iy)^3$.
31. $(a+ib)^2-(a-ib)^2$.
32. $(2+2\sqrt{-3})^3-(2-2\sqrt{-3})^3$.
33. $(a+i\sqrt{1-b^2})(a-i\sqrt{1-b^2})$.
34. Determine whether the sum and the product of the numbers $5+6\sqrt{-2}$ and $5-6\sqrt{-2}$ are real.
35. Determine whether the sum and the product of the numbers $2+\sqrt{-3}$ and $2-\sqrt{-3}$ are real.

94. Division of imaginaries. One complex number is the **conjugate** of another if their product and their sum are real. Thus $a+bi$ and $a-bi$ are conjugates. Conjugate

complex numbers are used in division of imaginary expressions as conjugate radicals are used in division of real radicals.

In case either the numerator or the denominator of a fraction is imaginary or complex, the division may be performed as in the following

EXAMPLES

1. $\sqrt{-6} \div \sqrt{2}$.

Solution. $$\frac{\sqrt{-6}}{\sqrt{2}} = \frac{\sqrt{-6}\cdot\sqrt{2}}{(\sqrt{2})^2} = \frac{2\sqrt{-3}}{2} = \sqrt{3}\,i.$$

2. $\sqrt{8} \div \sqrt{-2}$.

Solution. $$\frac{\sqrt{8}}{\sqrt{-2}} = \frac{\sqrt{8}\cdot\sqrt{-2}}{(\sqrt{-2})^2} = \frac{4i}{-2} = -2i.$$

3. $\sqrt{-6} \div \sqrt{-2}$.

Solution. $$\frac{\sqrt{-6}}{\sqrt{-2}} = \frac{\sqrt{-6}\cdot\sqrt{-2}}{(\sqrt{-2})^2} = \frac{\sqrt{6}\cdot i\cdot\sqrt{2}\cdot i}{-2} = \sqrt{3}.$$

4. $3 \div (2+\sqrt{-3})$.

Solution. $$\frac{3}{2+\sqrt{-3}} = \frac{3(2-\sqrt{-3})}{(2+\sqrt{-3})(2-\sqrt{-3})} = \frac{6-3\sqrt{-3}}{4+3}$$
$$= \frac{6-3\sqrt{-3}}{7}.$$

The method of the above examples is stated in the

Rule. *Write the dividend over the divisor in the form of a fraction.*

Then multiply both numerator and denominator of this fraction by the simplest expression which will make the new denominator real and rational.

Reduce the result to its simplest form.

EXERCISES

Perform the indicated operations:

1. $\sqrt{-12} \div \sqrt{2}$.
2. $\sqrt{6} \div \sqrt{-2}$.
3. $2\sqrt{5} \div 4\sqrt{-1}$.
4. $\sqrt{-9} \div \sqrt{-1}$.
5. $1 \div \sqrt{-3}$.
6. $4 \div \sqrt{-5}$.
7. $\sqrt{8} \div \sqrt{-2}$.
8. $(-49)^{\frac{1}{2}} \div (-64)^{\frac{1}{2}}$.
9. $\sqrt{by} \div \sqrt{-y}$.
10. $\sqrt{-m} \div \sqrt{-n}$.
11. $(-6bx)^{\frac{1}{2}} \div (-5x)^{\frac{1}{2}}$.
12. $\left[(-x^6)^{\frac{1}{2}} - (-x^2)^{\frac{1}{2}}\right] \div (-x)^{\frac{1}{2}}$.
13. $3 \div (1 + \sqrt{-1})$.
14. $2 \div (1 - \sqrt{-2})$.
15. $2\sqrt{-5} \div (\sqrt{-1} + 6)$.
16. $2\sqrt{-3} \div (3\sqrt{-2} + 3)$.
17. $(-1 - \sqrt{-3}) \div (-1 + \sqrt{-3})$
18. $(1 + 2i) \div (3 - 4i)$.
19. $x \div (x + iy)$.
20. $(a + ib) \div (c + id)$.
21. $(3 + 2i)(1 - i) \div (3 - 4i)(1 + i)$.
22. Is $+1 - \sqrt{-3}$ a cube root of -8?
23. Does $x^2 - 6x + 12 = 0$ if $x = 3 \pm \sqrt{-3}$?
24. Does $x = \frac{8}{5}(\sqrt{-10})$, $y = -\frac{3}{5}(\sqrt{-10})$, satisfy the system $x^2 - xy - 12y^2 = 8$, $x^2 + xy - 10y^2 = 20$?

95. Equations with imaginary roots. The student should now be able to check the solution of an equation which has imaginary roots.

EXERCISES

Solve the equations which follow, and check the results:

1. $x^2 + 4x + 8 = 0$.
2. $x^2 - 8x + 24 = 0$.
3. $x^2 + 3x + 9 = 0$.
4. $x^2 - 5x + 16 = 0$.
5. $3x^2 + 2x + 4 = 0$.
6. $x^2 + x + 1 = 0$.
7. $x^2 - x + 1 = 0$.
8. $5x^2 - 6x + 14 = 0$.
9. $6x^2 + 10x + 21 = 0$.
10. $3x^2 + 16x + 21 = 0$.

11. $x^3 = 1$.

Hints. If $x^3 = 1$, $\quad x^3 - 1 = 0.$

Hence $\quad (x-1)(x^2 + x + 1) = 0.$

Then $\quad x - 1 = 0,$

and $\quad x^2 + x + 1 = 0.$

12. $x^3 + 1 = 0$.
13. $x^3 = 8$.
14. $x^3 = -27$.
15. $x^4 = 1$.
16. $x^4 = 9$.
17. $x^6 = 1$.
18. $x^6 = 64$.
19. $x^3 = 64$.
20. $x^3 = -125$.

21. How many square roots has any real number? cube roots? fourth roots? sixth roots?

22. What do the preceding exercises suggest regarding the number of nth roots which any real number has?

23. $27x^3 - 8 = 0$.
24. $64x^3 + 125 = 0$.
25. $x^4 - 2x^2 - 8 = 0$.
26. $x^3 + x^2 - 2x - 2 = 0$.
27. $x^6 + 7x^3 - 8 = 0$.
28. $3x^4 + 16x^2 + 21 = 0$.
29. $27x^4 - 12x^2 - 64 = 0$.
30. $6x^6 + 21x^3 + 9 = 0$.
31. $25x^4 + 40x^2 + 64 = 0$.
32. $(x^2 + 4)(x^2 + 3x + 7) = 0$.
33. $(x^2 + 2x)^2 + 15(x^2 + 2x) + 54 = 0$.
34. $(x^2 + 5x)^2 + 9(x^2 + 5x) - 112 = 0$.

35. Solve $x + y = 4$, $x^2 - 3xy - y^2 = -39$, and check the results.

36. Solve $x + 2y = 4$, $y^2 - x = 0$, and check the results.

37. Solve $x^2 + y^2 = 4$, $x - y = 6$, and check the results.

Note. Long before the time of Gauss, mathematicians had performed the operations of multiplication and division on complex numbers by the same rules that they used for real numbers. As early as 1545 Cardan showed that the product of $5 + \sqrt{-15}$ and $5 - \sqrt{-15}$ was the real number 40. However, he was not always equally fortunate in obtaining correct results, for in another place he sets $\frac{1}{4}\left(-\sqrt{-\frac{1}{4}}\right) = \frac{1}{\sqrt{64}} = \frac{1}{8}$.

Even the rather complicated formula for extracting any root of a complex number was discovered in the early part of the eighteenth century. But all these operations were purely formal, and seemed to most mathematicians a mere juggling with symbols until Gauss showed clearly the place and usefulness of such numbers.

96. Factors involving imaginaries. After studying radicals we enlarged our previous notion of a factor and, with certain limitations, employed radicals among the terms of a factor. Now in a similar manner, with like restrictions, we extend our notion of a factor still farther and use imaginary numbers as coefficients or as terms in a factor. For example, x^2+1 may hereafter be regarded as factorable,

for $$x^2+1=x^2-(-1)=(x+\sqrt{-1})(x-\sqrt{-1}).$$

Similarly,

$$4x^2+9=4x^2-(-9)=(2x+3\sqrt{-1})(2x-3\sqrt{-1})$$

and $$x^2+6=x^2-(-6)=(x+\sqrt{-6})(x-\sqrt{-6}).$$

Further, $x^3-1=(x-1)(x^2+x+1)$. Hitherto the trinomial x^2+x+1 has been regarded as prime; but the student can easily prove that x^2+x+1 is equal to the product $(x+\frac{1}{2}+\frac{1}{2}\sqrt{-3})(x+\frac{1}{2}-\frac{1}{2}\sqrt{-3})$. Therefore x^3-1 has three factors, these two and $x-1$.

Note on the use of imaginaries. We have explained the laws of addition, subtraction, multiplication, and division for imaginary (and complex) numbers and have made some use of them. It is largely because imaginaries obey these laws that we call them numbers, for it must be admitted that we cannot count objects with imaginary numbers. Nor can we state by means of them our age, our weight, or the area of the earth's surface. It should be remembered, however, that we can do none of these things with negative numbers. We may have a group of objects — books, for example — whose number is 5; but no group of *objects* exists whose number is -5, or -3, or any negative number whatever. If it is asked, How, then, can negative numbers and imaginary numbers

have any practical use? the answer is this: They have a practical use because when they enter into our calculations and we have performed the necessary operations upon them and obtained our final result, that result can frequently be interpreted as a concrete number like those dealt with in ordinary arithmetic. Moreover, if the result cannot be so interpreted, it is, in applied mathematics at least, finally rejected.

In that part of electrical engineering where the theory and measurement of alternating currents of electricity are treated, complex numbers have had extensive use. Their employment in the difficult problems which there arise has given a briefer, a more direct, and a more general treatment than the earlier ones where such numbers are not used.

In theoretical mathematics complex numbers have been of great value in many ways. For example, numerous important theorems about functions are more easily proved under the assumption that the variable is complex. Then, by letting the imaginary part of the complex number become zero, we obtain the proof of the theorem for real values of the variable. Indeed, the student need not go very far beyond this point in his mathematical work to learn that, if e is $2.7182+$ (see page 253), $e^{\sqrt{-1}} + e^{-\sqrt{-1}}$ is equal to the real number $1.082+$. At the same time he will learn also how such a form arises, and something of its importance. In a way which we cannot now explain, even so involved an expression as $(a+ib)^{c+id}$ has in higher work a meaning and a use. If the student pursues his mathematical studies far enough, that meaning and use and a multitude of other uses for complex numbers will become familiar to him. But the numbers which we have learned in this book to use, namely fractions, negative numbers, irrational numbers, and complex numbers, complete the number system of ordinary algebra, for it can be proved that from the fundamental operations no other forms of number can arise.

CHAPTER XIII

THEORY OF QUADRATIC EQUATIONS

97. Formation of equations with given roots. According to section 34, the equation $(x-2)(x-3)=0$ has the roots 2 and 3. In general, the equation $(x-r_1)(x-r_2)=0$ has the roots r_1 and r_2, because either of these numbers, when substituted for x, satisfies the equation. Hence we can always find an equation whose roots are two given numbers r_1 and r_2 by setting the product of the binomials $x-r_1$ and $x-r_2$ equal to zero.

For example, an equation whose roots are 3 and 4 is seen in $(x-3)(x-4)=0$, or $x^2-7x+12=0$.

EXERCISES

Form an equation whose roots are the following:

1. 2, 3.

2. 3, 7.

3. 1, -3.

4. -2, -5.

5. 2, $\frac{2}{5}$.

6. 5, $\frac{3}{7}$.

7. $-\frac{2}{3}$, $\frac{5}{6}$.

8. $-\frac{4}{5}$, $-\frac{7}{8}$.

9. $1+\sqrt{3}$, $1-\sqrt{3}$.

10. $3 \pm \sqrt{7}$.

11. $2+\sqrt{-5}$, $2-\sqrt{-5}$.

12. $-7+\sqrt{-5}$, $-7-\sqrt{-5}$.

13. $\frac{1}{2}+\sqrt{\frac{3}{2}}$, $\frac{1}{2}-\sqrt{\frac{3}{2}}$.

14. $\frac{1}{2}+\sqrt{-\frac{3}{2}}$, $\frac{1}{2}-\sqrt{-\frac{3}{2}}$.

15. 2, 3, 4.

Hint. $(x-2)(x-3)(x-4)=0$.

16. 1, 3, 5.

17. 1, -1, 2.

18. $\frac{1}{2}$, $\frac{1}{3}$, $\frac{1}{4}$.

19. $\sqrt{-2}$, $-\sqrt{-2}$, 2.

20. $1+\sqrt{2}$, $1-\sqrt{2}$, 3.

21. 1, -1, $\sqrt{-1}$, $-\sqrt{-1}$.

22. $\sqrt{2}-\sqrt{3}$, $\sqrt{2}+\sqrt{3}$, $-\sqrt{2}-\sqrt{3}$, $-\sqrt{2}+\sqrt{3}$.

98. Relations between roots and coefficients. By direct multiplication we obtain from

$$(x-r_1)(x-r_2)=0 \qquad (1)$$

the equation $$x^2-(r_1+r_2)x+r_1r_2=0. \qquad (2)$$

Since r_1 and r_2 are the roots of (1), it appears from an inspection of (2) that the quadratic equation

$$x^2+bx+c=0$$

has the roots r_1 and r_2, provided $b=-(r_1+r_2)$ and $c=r_1r_2$.

For example, we may form at once the equation whose roots are 4 and 9, as follows:

$$x^2-(4+9)x+4\cdot 9=0, \text{ or } x^2-13x+36=0.$$

Similarly for the cubic equation $x^3+bx^2+cx+d=0$, whose roots are r_1, r_2, and r_3, we have $(x-r_1)(x-r_2)(x-r_3)=0$.

Then $$b=-(r_1+r_2+r_3),$$
$$c=r_1r_2+r_1r_3+r_2r_3,$$
and $$d=-r_1r_2r_3.$$

ORAL EXERCISES

Form equations whose roots are the following:

1. 2, 9.
2. 4, 5.
3. -1, 6.
4. -3, -5.
5. -7, 2.
6. $\sqrt{3}$, $-\sqrt{3}$.
7. $2\sqrt{2}$, $-2\sqrt{2}$.
8. $3+\sqrt{7}$, $3-\sqrt{7}$.
9. $1+\sqrt{2}$, $1-\sqrt{2}$.

We will now show the relations which exist between the roots and the coefficients of the general quadratic equation $ax^2+bx+c=0$. By section 88 the roots of $ax^2+bx+c=0$ are

$$r_1=\frac{-b+\sqrt{b^2-4ac}}{2a} \quad \text{and} \quad r_2=\frac{-b-\sqrt{b^2-4ac}}{2a}.$$

Adding r_1 and r_2, we have

$$r_1 + r_2 = \frac{-b+\sqrt{b^2-4ac}-b-\sqrt{b^2-4ac}}{2a} = \frac{-2b}{2a} = -\frac{b}{a}.$$

Therefore $$-(r_1+r_2) = \frac{b}{a}.$$

Multiplying r_1 by r_2, we have

$$r_1 r_2 = \frac{-b+\sqrt{b^2-4ac}}{2a}\cdot\frac{-b-\sqrt{b^2-4ac}}{2a}$$

$$= \frac{b^2-(b^2-4ac)}{4a^2} = \frac{4ac}{4a^2} = \frac{c}{a}.$$

Therefore $$r_1 r_2 = \frac{c}{a}.$$

These results may be expressed verbally as follows:

For the equation $ax^2 + bx + c = 0$,

I. The sum of the roots with its sign changed is $\frac{b}{a}$.

II. The product of the roots is $\frac{c}{a}$.

These relations frequently afford the simplest means of checking the result of solving a quadratic equation, as illustrated in Exercises 18–31, below.

EXERCISES

Form the equation whose roots are the following:

1. $\frac{3}{2}$, $-\frac{4}{5}$.

Solution. $-\left(\frac{3}{2}-\frac{4}{5}\right) = -\frac{7}{10} = \frac{b}{a}$; $\left(\frac{3}{2}\right)\left(-\frac{4}{5}\right) = -\frac{6}{5} = \frac{c}{a}$.

Hence the required equation is

$$x^2 - \frac{7}{10}x - \frac{6}{5} = 0, \text{ or } 10x^2 - 7x - 12 = 0.$$

2. $\frac{4}{3}, \frac{1}{2}$.

3. $\frac{1}{4}, -\frac{2}{3}$.

4. $1\frac{1}{2}, -3\frac{1}{5}$.

5. $4.41, 1.59$.

6. $2+3\sqrt{3}, 2-3\sqrt{3}$.

7. $4+\sqrt{-2}, 4-\sqrt{-2}$.

8. $\frac{3}{2}+\sqrt{7}, \frac{3}{2}-\sqrt{7}$.

9. $\frac{1}{2}+\frac{1}{2}\sqrt{-3}, \frac{1}{2}-\frac{1}{2}\sqrt{-3}$.

10. $1\frac{2}{3}\pm\sqrt{3}$.

11. $\pm\sqrt{-1}+1$.

12. $a, -\frac{1}{a}$.

13. $\frac{3a}{2}, \frac{5a}{2}$.

14. $1+a, 1-a$.

15. $\frac{a+b}{a-b}, \frac{a-b}{a+b}$.

16. $6, 8, \frac{1}{2}$.

17. $-4, 4, \frac{1}{4}$.

Solve the following equations by the use of the formula and check the result by the use of I and II, above:

18. $x^2-5x+6=0$

19. $x^2-x-3=0$.

20. $x^2-2x-4=0$.

21. $x^2-9x-10=0$.

22. $x^2+2x+1=0$.

23. $x^2+8x+16=0$.

24. $x^2+5x+5=0$.

25. $2x^2+3x-6=0$.

26. $3x^2+3x-5=0$.

27. $5x^2-6x+10=0$.

28. $x^2+x+1=0$.

29. $\frac{x^2}{3}+\frac{x}{4}+\frac{1}{5}=0$.

30. $\frac{x^2}{2}+4x-7=0$.

31. $6x^2-4x-3=0$.

Find the value of the literal coefficient in the following:

32. $x^2+2x-c=0$ if one root is 2.

33. $x^2-x-c=0$, if one root is 6.

34. $x^2-cx-70=0$, if one root is 7.

35. $x^2+2bx+20=0$, if one root is -4.

36. $x^2-8x+c=0$, if one root is twice the other.

37. $x^2+7x+c=0$, if one root exceeds the other by 2.

38. $x^2+11x+b=0$, if the difference between the roots is 10.

99. Character of the roots of a quadratic equation. It is often desirable to determine whether the roots of a given quadratic equation are real or imaginary, rational or irrational, equal or unequal, without solving the equation. This can be accomplished by use of the formulas for the roots of the quadratic $ax^2 + bx + c = 0$:

$$r_1 = \frac{-b + \sqrt{b^2 - 4ac}}{2a}, \tag{1}$$

$$r_2 = \frac{-b - \sqrt{b^2 - 4ac}}{2a}. \tag{2}$$

These expressions are seen to differ from each other only in the sign preceding the radical. The expression

$$b^2 - 4ac,$$

which appears under the radical sign, is called the **discriminant** of the quadratic. The only way in which r_1 or r_2 can be a complex number is for the discriminant to be negative. If all of the coefficients a, b, and c are rational, r_1 or r_2 can be rational only when $\sqrt{b^2 - 4ac}$ is rational; that is, when the discriminant is a perfect square.

Hence if a, b, and c are rational, an inspection of (1) and (2) shows that the following statements are true:

I. If $b^2 - 4ac$ is positive and not a perfect square, the roots are real, unequal, and irrational.

For example, in $x^2 - 8x + 2 = 0$ the discriminant $b^2 - 4ac$ equals $(-8)^2 - 4 \cdot 1 \cdot 2 = 56$, which is not a perfect square. The roots of the equation are the real, unequal, and irrational numbers $4 \pm \sqrt{14}$.

II. If $b^2 - 4ac$ is positive and a perfect square, the roots are real, unequal, and rational.

For example, in the equation $2x^2 - 3x - 9 = 0$ the discriminant $b^2 - 4ac$ equals $(-3)^2 - 4 \cdot 2 \cdot (-9) = 81$, which is a perfect square. The roots are the real, unequal, rational numbers $-\frac{3}{2}$ and $+3$.

III. If $b^2 - 4\,ac$ is zero, the roots are equal.

For example, in the equation $4\,x^2 - 12\,x + 9 = 0$, the discriminant $b^2 - 4\,ac$ equals $(-12)^2 - 4 \cdot 4 \cdot 9 = 144 - 144 = 0$. The only number which satisfies this equation is $\frac{3}{2}$, so in one sense the equation has only one root. But since the left member has two identical factors each of which affords the same root of the equation, it is customary to say that the equation has equal roots.

IV. If $b^2 - 4\,ac$ is negative, the roots are imaginary.

For example, in the equation $2\,x^2 - 5\,x + 4 = 0$ the discriminant $b^2 - 4\,ac$ equals $(-5)^2 - 4 \cdot 2 \cdot 4 = 25 - 32 = -7$. The roots of the equation are the conjugate imaginaries $\dfrac{+5 + \sqrt{-7}}{4}$ and $\dfrac{+5 - \sqrt{-7}}{4}$.

ORAL EXERCISES

Determine the character of the roots of the following equations without solving:

1. $x^2 + 5\,x + 6 = 0$.
2. $3\,x^2 - 4\,x + 1 = 0$.
3. $2\,x^2 - 6\,x - 3 = 0$.
4. $2\,x^2 - 3\,x - 2 = 0$.
5. $5\,x^2 - 5\,x + 4 = 0$.
6. $x^2 + x + 1 = 0$.
7. $x^2 - 8\,x + 16 = 0$.
8. $2\,x^2 + 3\,x + 5 = 0$.
9. $4\,x^2 - 4\,x + 1 = 0$.
10. $3\,x^2 - 2\,x - 2 = 0$.

EXERCISES

Determine the value of k which will make the roots of the following equations equal:

1. $x^2 - kx + 16 = 0$.

Solution. $a = 1$, $b = -k$, $c = 16$.

Hence $$b^2 - 4\,ac = k^2 - 64.$$

In order for the roots to be equal, $b^2 - 4\,ac$ must equal zero.

Therefore $$k^2 - 64 = 0.$$

Whence $$k = \pm 8.$$

Check. Substituting 8 for k in the original equation,

$$x^2 - 8x + 16 = 0.$$

Whence $x = 4$, only.

Similarly, substituting -8 for k,

$$x^2 + 8x + 16 = 0.$$

Whence $x = -4$, only.

2. $x^2 + kx + 16 = 0.$
3. $x^2 - 10x + k = 0.$
4. $2x^2 + 8x + k = 0.$
5. $x^2 - 3kx + 36 = 0.$
6. $3x^2 + 4kx + 12 = 0.$
7. $9x^2 + 30x + k + 9 = 0.$
8. $4kx^2 - 60x + 25 = 0.$
9. $9k^2x^2 - 84x + 49 = 0.$
10. $49x^2 - (k+3)x + 4 = 0.$
11. $(k^2+5)x^2 - 30x + 25 = 0.$

Determine the values of a for which the following systems will have two sets of equal roots:

12. $\begin{cases} y^2 = 2x, \\ y = x + a. \end{cases}$
13. $\begin{cases} x^2 + y^2 = a^2, \\ y = x + 1. \end{cases}$
14. $\begin{cases} xy = a, \\ x + y = 1. \end{cases}$

100. Number of roots of a quadratic. In section 87 we found that every quadratic equation has two roots.

Up to the present we have assumed that a quadratic equation can have no other root than the ones found by the method of completing the square. This fact can be proved as follows:

Proof. If we write the equation $ax^2 + bx + c = 0$ in the form $x^2 + \frac{b}{a} \cdot x + \frac{c}{a} = 0$ and substitute therein from (I) and (II) on page 170, we get $x^2 - (r_1 + r_2)x + r_1r_2 = 0$. This can be factored and written as $(x - r_1)(x - r_2) = 0$. Now if any value of x different from r_1 and r_2, say r, be a root of this equation, such a value when substituted for x must satisfy the equation $(x - r_1)(x - r_2) = 0$.

Hence $(r - r_1)(r - r_2)$ must equal zero. By assumption, however, r is different from r_1 and r_2. Consequently neither the factor $r - r_1$

nor $r - r_2$ can equal zero, and therefore their product cannot equal zero. This proves that no additional value r can satisfy the equation $x^2 - (r_1 + r_2)x + r_1r_2 = 0$. As this equation is but another form of $ax^2 + bx + c = 0$, the latter has only two roots.

101. Factors of quadratic expressions. Let r_1 and r_2 be the roots of $ax^2 + bx + c = 0$; then

$$x^2 + \frac{b}{a}x + \frac{c}{a} = x^2 - (r_1 + r_2)x + r_1r_2 = (x - r_1)(x - r_2),$$

or
$$ax^2 + bx + c = a(x - r_1)(x - r_2).$$

Therefore the three factors a, $x - r_1$, and $x - r_2$ of any quadratic expression can be found if we first set the expression equal to zero (see section 34) and solve the equation thus formed. Obviously the character of the roots so obtained will determine the character of the factors. Hence by the use of the discriminant $b^2 - 4ac$ we can decide whether the factors of a quadratic expression are real or imaginary, rational or irrational, without factoring it.

EXERCISES

Determine which of the following expressions have rational factors:

1. $x^2 - 3x - 40$.
2. $2x^2 + 5x - 7$.
3. $7x^2 - 9x + 18$.
4. $24x^2 - x - 10$.
5. $72x^2 - 17x + 1$.
6. $5x^2 + 3x - 20$.
7. $3x^2 - 9x + 28$.
8. $33h^2 - 233h - 6$.
9. $x^2 - 2ax + (a^2 - b^2)$.
10. $abx^2 - (b^2 + a^2)x + ab$.

Separate into rational, irrational, or imaginary factors:

11. $2x^2 + 5x - 8$.

Solution. Let $2x^2 + 5x - 8 = 0$.

Solving by formula, $x = \dfrac{-5 \pm \sqrt{25 - (-64)}}{4} = \dfrac{-5 \pm \sqrt{89}}{4}$.

Then $\qquad r_1 = \dfrac{-5+\sqrt{89}}{4}$ and $r_2 = \dfrac{-5-\sqrt{89}}{4}$.

Therefore $2x^2+5x-8 = 2\left[x - \dfrac{-5+\sqrt{89}}{4}\right]\left[x - \dfrac{-5-\sqrt{89}}{4}\right]$

$$= \tfrac{1}{8}(4x+5-\sqrt{89})(4x+5+\sqrt{89}).$$

12. x^2-7x-7.

13. x^2-4x-1.

14. x^2+2x+2.

15. x^2+4x-9.

16. $4x^2-12x-9$.

17. $25x^2+20x+4$.

18. $6x^2+14x-40$.

19. $10-9x-9x^2$.

20. $10x^2+12-26x$.

21. x^2+7x+8.

22. x^2+x+1.

23. x^2+1.

24. x^2+9.

25. $x^2-2ax+a^2-b$.

26. $x^2+6ax+9a^2-4b$.

27. $4x^2+4ax+a^2-4c$.

28. $x^2-4ax+4a^2+c$.

29. ax^2+bx+c.

30. $x^2-xy+5x-2y+6$.

Solution. Let $\qquad x^2-xy+5x-2y+6=0$.

Then $\qquad x^2+(5-y)x-2y+6=0$.

Solving for x in terms of y by the formula,

$$x = \frac{-(5-y)\pm\sqrt{(5-y)^2-4(-2y+6)}}{2}$$

$$= \frac{-5+y\pm\sqrt{y^2-2y+1}}{2}.$$

Whence $\qquad x=-2$ and $y-3$.

Therefore $x^2-xy+5x-2y+6=(x+2)(x-y+3)$.

31. $3x^2-6xy+14x-4y+8$.

32. $x^2-xy-2y^2+3x-6y$.

33. $x^2-4xy-y+3y^2-2-x$.

CHAPTER XIV

GRAPHS OF QUADRATIC EQUATIONS IN TWO VARIABLES

102. Graph of a quadratic equation in two variables. Before solving graphically a quadratic system, the method of graphing *one* quadratic equation in *two* variables must be clearly understood.

EXAMPLES

1. Construct the graph of $x^2 = 3y$.

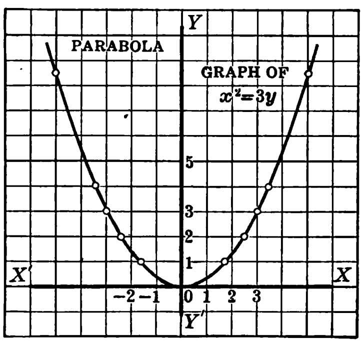
Solution. Solving the equation for y, $y = \frac{x^2}{3}$.

We now assign values to x and then compute the approximate corresponding values of y. Tabulating the results gives:

If $x =$	6	4	3	2	1	0	-1	-2	-4	-6
then $y =$	12	$\frac{16}{3}$	3	$\frac{4}{3}$	$\frac{1}{3}$	0	$\frac{1}{3}$	$\frac{4}{3}$	$\frac{16}{3}$	12

Using an x-axis and a y-axis as in graphing linear equations, plotting the points corresponding to the real numbers in the table, and drawing the curve determined by these points, we obtain the graph of the adjacent figure. The curve is called **a parabola.**

The graph of any equation of the form $x^2 = ay$ is a parabola.



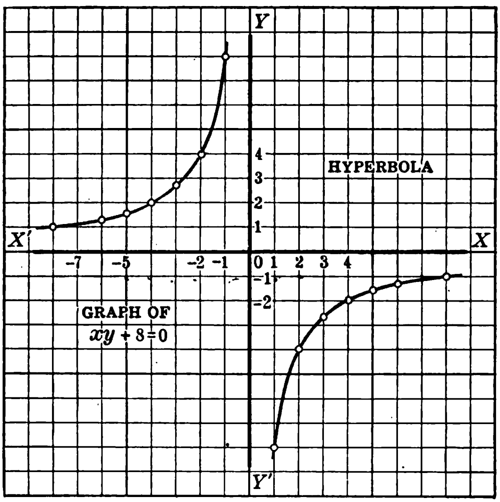
2. Graph the equation $xy + 8 = 0$.

Solution. Solving for y, $y = -\dfrac{8}{x}$.

Assigning values to x as indicated in the following table, we then compute the corresponding values of y:

If $x =$	-6	-5	-4	-3	-2	-1	$-\frac{1}{2}$	$\frac{1}{2}$	1	2	3	4	5	6	8
then $y =$	$\frac{4}{3}$	$\frac{8}{5}$	2	$\frac{8}{3}$	4	8	16	-16	-8	-4	$-\frac{8}{3}$	-2	$-\frac{8}{5}$	$-\frac{4}{3}$	-1

Proceeding as before with the numbers in the table, we obtain the two-branched curve of the figure below, which does not touch either axis. The curve is called **a hyperbola.**



The graph of any equation of the form $xy = K$ is a *hyperbola.* The curve for $xy = K$ ($K =$ any constant) is always in the same general position; that is, if K is positive, one branch of the curve lies in the first quadrant and the other branch in the third. If K is negative, one branch lies in the second quadrant and the other in the fourth.

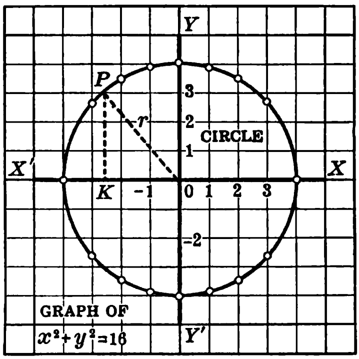
3. Graph the equation $x^2 + y^2 = 16$.

Solution. Solving for y, $y = \pm\sqrt{16 - x^2}$.

Assigning values to x as indicated in the following table, we obtain from page 274 the corresponding values of y:

If $x=$	-5	-4	-3	-2	-1	0	1	2	3	4	5
then $y=$	$\pm 3\sqrt{-1}$	0	± 2.64	± 3.46	± 3.87	± 4	± 3.87	± 3.46	± 2.64	0	$\pm 3\sqrt{-1}$

For values of x numerically greater than 4 it appears that y is imaginary. The points corresponding to the pairs of real numbers in the table lie on the circle in the accompanying figure. The center of the circle is at the origin, and the radius is 4.



The graph of any equation of the form $x^2 + y^2 = r^2$ is a circle whose radius is r. This can be proved from the right triangle PKO. If P represents *any* point on the circle, OK equals the x-distance of P, KP equals the y-distance, and OP equals the radius. Now $\overline{OK}^2 + \overline{KP}^2 = \overline{OP}^2$; that is, $x^2 + y^2 = r^2$. It follows, then, that the graphs of $x^2 + y^2 = 9$ and $x^2 + y^2 = 8$ are circles whose centers are at the origin and whose radii are 3 and $\sqrt{8}$ respectively. Hereafter, when it is required to graph an equation of the form $x^2 + y^2 = r^2$, the student may use compasses and, with the origin as the center and with the proper radius (the square root of the constant term), describe the circle at once.

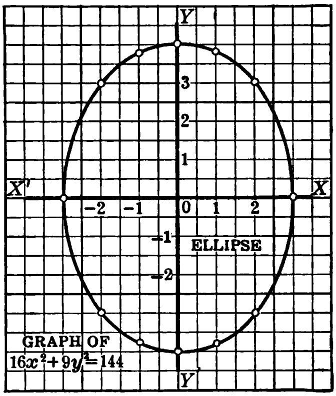
In all the graphical work which follows, the student will save time by obtaining from the table on page 274 the square roots or cube roots which he may need.

4. Graph the equation $16x^2 + 9y^2 = 144$.

Solution. Solving, $y = \pm \frac{4}{3}\sqrt{9 - x^2}$. Proceed as in Example 3:

If $x =$	-4	-3	-2	-1	0	$+1$	$+2$	$+3$	$+4$
then $y =$	$\pm\frac{4}{3}\sqrt{-7}$	0	± 2.98	± 3.77	± 4	± 3.77	± 2.98	0	$\pm\frac{4}{3}\sqrt{-7}$

For any value of x numerically greater than 3, y is imaginary. The points corresponding to the real numbers in the table lie on the graph of the adjacent figure. The curve is called an **ellipse.**



The graph of any equation of the form of $ax^2 + by^2 = c$ in which a and b are unequal and of the same sign as c is an *ellipse.*

NOTE. These three curves — the ellipse, the hyperbola, and the parabola — were first studied by the Greeks, who proved that they are the sections which one obtains by cutting a cone by a plane. Not for hundreds of years afterwards did anyone imagine that these curves actually appear in nature, for the Greeks regarded them merely as geometrical figures, and not at all as curves that have anything to do with our everyday life. One of the most important discoveries of astronomy was made by Kepler (1571–1630), who showed that the earth revolves around the sun in an ellipse, and stated the laws which govern the motion. Those comets that return to our field of vision periodically also have elliptic orbits, while those that appear once, never to be seen again, describe parabolic or hyperbolic paths.

The path of a ball thrown through the air in any direction, except vertically upward or downward, is a parabola. The approximate parabola which a projectile actually describes depends on the elevation of the gun (the angle with the horizontal), the quality of the powder, the amount of the charge, the direction of the wind, and various other conditions. This makes gunnery a complex subject.

EXERCISES

Construct the graphs of the following equations and state the name of each curve obtained:

1. $x^2 = 4y$.
2. $y^2 + 2x = 0$.
3. $x^2 + y^2 = 49$.
4. $x^2 + y^2 = 18$.
5. $x^2 - y^2 = 16$.
6. $xy = 12$.
7. $xy = -6$.
8. $9x^2 + 4y^2 = 36$.
9. $16x^2 - 9y^2 = 144$.
10. $25x^2 + 9y^2 = 225$.

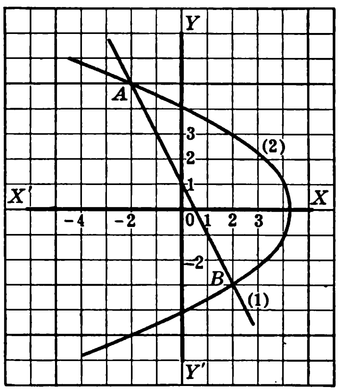
103. Graphical solution of a quadratic system in two variables. That we may solve a system of two quadratic equations by a method similar to that employed in section 44 for linear equations appears from the

EXAMPLES

1. Solve graphically $\begin{cases} 2x + y = 1, & (1) \\ y^2 + 4x = 17. & (2) \end{cases}$

Solution. Constructing the graphs of (1) and (2), we obtain the straight line and the parabola shown in the adjacent figure. There are two sets of roots corresponding to two points of intersection, which are

$$A\begin{cases} x = -2, \\ y = 5. \end{cases} \qquad B\begin{cases} x = 2, \\ y = -3. \end{cases}$$

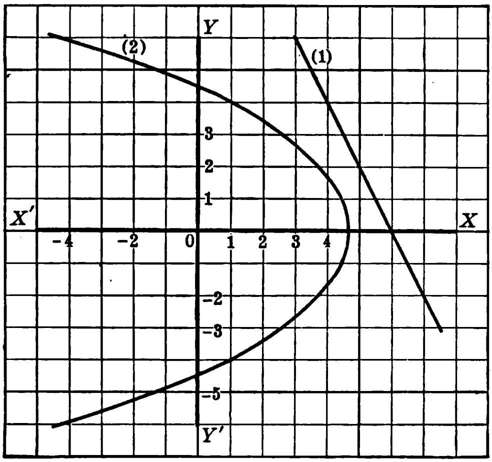


NOTE. If the straight line in the adjacent figure were moved to the right in such a way that it always remained parallel to its present position, the points A and B would approach each other and finally coincide. The line would then be tangent to the parabola at the point $x = 4$, $y = 1$.

Were the straight line moved still farther, it would neither touch nor intersect the parabola and there would be no graphical solution (see page 182).

2. Solve graphically $\begin{cases} 2x + y = 12, & (1) \\ y^2 + 4x = 19. & (2) \end{cases}$

Solution. The graphs of (1) and (2) are the straight line and the parabola of the adjacent figure. These curves have no real points of intersection. There are, however, two pairs of imaginary roots. Solving (1) and (2) by substitution, $x = \frac{11}{2} + \sqrt{-1}$ or $\frac{11}{2} - \sqrt{-1}$, and $y = 1 - 2\sqrt{-1}$ or $1 + 2\sqrt{-1}$.



The essential point to be emphasized here is that real roots of a simultaneous system correspond to real intersections, and imaginary roots correspond to no intersections of real graphs.

3. Solve graphically $\begin{cases} x^2 - y^2 = 4, & (1) \\ y^2 - x - 6 = 0. & (2) \end{cases}$

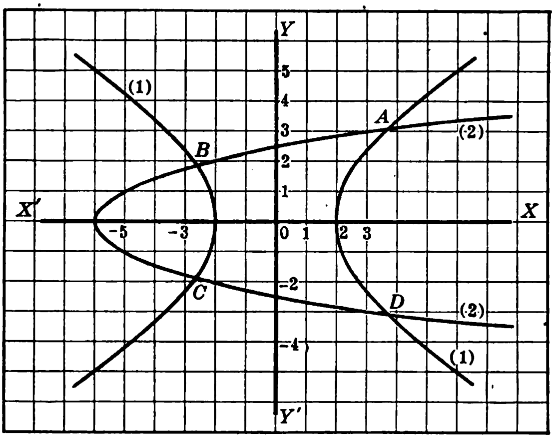
Solution. Constructing the graphs of (1) and (2), we obtain the hyperbola and the parabola of the figure on the opposite page. There are four sets of roots corresponding to the four points of intersection, which are approximately

$$A\begin{cases} x = 3.7, \\ y = 3.1. \end{cases} \quad B\begin{cases} x = -2.7, \\ y = 1.8. \end{cases} \quad C\begin{cases} x = -2.7, \\ y = -1.8. \end{cases} \quad D\begin{cases} x = 3.7, \\ y = -3.1. \end{cases}$$

If the two curves had been so chosen as to intersect only twice, their equations would have had only two sets of real roots.

Examples 1, 2, and 3 partially illustrate the truth of the following statement:

If in a system of two equations in two variables one equation is of the mth degree and one of the nth, there are *usually* mn sets of roots (real or imaginary) *and never more than* mn *such sets.*



EXERCISES

If possible, solve graphically each of the following systems:

1. $y^2 = 4x,$ $3x + y = 5.$

2. $x^2 + y^2 = 25,$ $y - 2x = 10.$

3. $xy = 6,$ $3x + y = 9.$

4. $x^2 + y^2 = 25,$ $x^2 + y^2 = 16.$

5. $x^2 + y^2 = 9,$ $x + y = 10.$

6. $x^2 + y^2 = 36,$ $x^2 - y^2 = 25.$

7. $x^2 + y^2 = 9,$ $x^2 - y^2 = 16.$

8. $xy = 12,$ $2x + y = 10.$

9. $x^2 = 4y,$ $9x^2 + y^2 = 9.$

10. $x^2 + 4y = 17,$ $x + 2y = 9.$

11. $x^2 + 4y = 17,$ $x + 2y = 12.$

12. $x^2 + y = 5,$ $y^2 + x = 3.$

13. $x - y\sqrt{8} = 0,$ $x^2 = y^3 - 9y.$

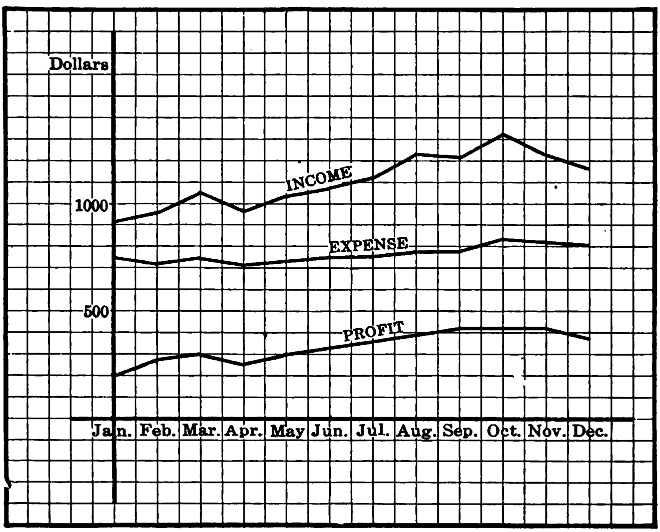
104. Graphical presentation of numerical data. A great variety of statistics can be presented graphically in a very striking manner. Business and commercial houses have during the past few years used the method extensively not only to present facts but also to aid in interpreting them and in indicating their tendencies.

The following exercises illustrate some of the possibilities in the graphical presentation of numerical data.

EXAMPLE

For the year 1912 the total income and expenses per mile of line of all the railroads in the United States having a yearly revenue of one million dollars or more was as follows:

	Jan.	Feb.	Mar.	Apr.	May	June	July	Aug.	Sept.	Oct.	Nov.	Dec.
Income in dollars .	930	970	1050	980	1040	1080	1120	1220	1210	1320	1220	1170
Expenses in dollars	730	710	750	720	740	750	760	780	780	830	810	800



The foregoing graph represents the given data on the same axes.

The lowest curve on the diagram shows the profits of the railroads. It was obtained by plotting the differences of the numbers in the table.

EXERCISES

1. The average incomes of 155 members of a certain college class for the first ten years after their graduation is given in the following table:

Years after graduation	1	2	3	4	5	6	7	8	9	10
Income in dollars. . .	706	902	1199	1651	2039	2408	2382	2709	3222	3804

Graph the data. What tendencies do you note?

2. The number of inches of rainfall during the month of July and the number of bushels of corn yielded per acre for a term of years in a certain locality is given in the following table:

Year	'89	'90	'91	'92	'93	'94	'95	'96	'97	'98	'99	'00	'01	'02
Rainfall	5.4	2.6	5.1	3.7	3.4	1.9	4.8	5.4	3.7	4.3	4.6	4.7	1.2	6.0
Corn yield	32	23	27	27	24	18	30	38	25	26	28	30	19	32

Plot these data on the same axes. How do you account for the similarity of the curves?

3. The numbers of hundreds of telephone calls in certain business and residential sections of New York City are given in the following table for various hours of the day. Plot both sets of data on the same axes and explain the reason for differences in the shapes of the graphs.

Time of day	7	8	9	10	11	Noon	1	2	3	4	5	6	7	8	9	10	11	12
Business district .	1	2	4	55	108	101	78	75	93	87	67	35	8	2	1.5	1	1	1
Residence district	.25	8	35	73	76	65	52	48	47	40	38	38	36	32	25	15	4	3

4. Measurements of the breadth of the heads of a thousand students in a certain school were as follows:

Head breadth in inches	5.5	5.6	5.7	5.8	5.9	6.0	6.1	6.2	6.3	6.4	6.5	6.6	6.7	6.8
Number of students . .	3	12	43	80	131	236	185	142	99	37	15	12	3	2

Construct a graph of the above data.

5. The chest measurements of 10,000 soldiers were tabulated as follows:

Chest measurement in inches	33	34	35	36	37	38	39	40	41	42	43	44	45	46	47	48
Number of soldiers	5	31	141	322	732	1305	1867	1882	1628	1148	645	160	87	38	7	2

Construct a graph of the above data.

It may appear accidental that the foregoing measurements should group themselves with any regularity. But if the number of measurements of this type is large and each is made with care, they obey a law called the law of probability. In fact the graph of the data in Exercises 4 and 5 is a close approximation to what is called the probability curve.

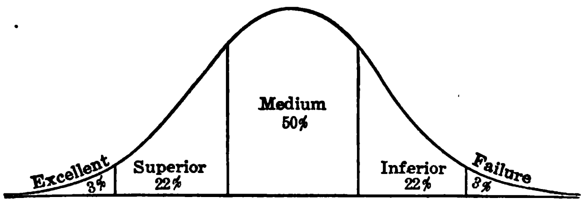
The equation of this curve is $y = e^{-x^2}$ when $e = 2.7$ approximately.

6. Construct the graph of the equation $y = e^{-x^2}$ between the values $x = -2$ and $x = 2$.

HINT. Let $x = -2, -\frac{3}{2}, -1, -\frac{1}{2}, 0$, etc.

NOTE. It is well established that physical characteristics, such as those illustrated by the graphs of Exercises 4 and 5, obey the law of probability. If the graph of Exercise 4 is carefully considered, it may raise the question, Do mental characteristics also obey the law? An interesting aspect of this is given by the fact that an increasing number of high schools and colleges assume that such is the case, and grade, or mark, their students by a system based

on the law of probability. Such a system assumes that if one hundred or more students in any subject are examined, the number of students and their degree of mastery of the subject arrange themselves according to the probability curve shown below. Obviously between very many students the differences in grades attained will be small, between many others they will be moderate, and between only a few will they be great.



In the statistical study of problems which have their origin quite remote from each other, this curve frequently occurs, and occupies a central position in the mathematical theory of statistics.

CHAPTER XV

SYSTEMS SOLVABLE BY QUADRATICS

105. Introduction. The general equation of the second degree in two variables is $ax^2 + by^2 + cxy + dx + ey + f = 0$. To solve a pair of such equations requires the solution of an equation of the fourth degree. Even the solution of $x^2 + y = 5$ and $y^2 + x = 3$ requires the solution of such an equation. In fact only a limited number of systems of the second degree in two variables is solvable by quadratics. By the graphical methods of Chapter XIV the student can solve graphically for real roots any system of quadratic equations, provided the terms have numerical coefficients. The algebraic solution of such systems will in many cases be possible for him only after further study of algebra.

106. Linear and quadratic systems. Every system of equations in two variables in which one equation is *linear* and the other *quadratic* can be solved by the method of *substitution.*

EXAMPLE

Solve the system $\begin{cases} x^2 - 2xy = 7, & (1) \\ x + 2y = 13. & (2) \end{cases}$

Solution. Solving (2) for y in terms of x,

$$y = \frac{13 - x}{2}. \qquad (3)$$

Substituting $\frac{13 - x}{2}$ for y in (1),

$$x^2 - 2x\left(\frac{13 - x}{2}\right) = 7. \qquad (4)$$

From (4), $$2x^2 - 13x - 7 = 0. \qquad (5)$$

Solving (5), $x = 7$ or $-\frac{1}{2}$.

Substituting 7 for x in (3),

$$y = \frac{13 - 7}{2}$$
$$= 3.$$

Substituting $-\frac{1}{2}$ for x in (3),

$$y = \frac{13 + \frac{1}{2}}{2}$$
$$= 6\tfrac{3}{4}.$$

The two sets of roots are $x = 7$, $y = 3$, and $x = -\frac{1}{2}$, $y = 6\frac{3}{4}$.

Check. Substituting 7 for x and 3 for y in (1) and (2),

$$49 - 42 = 7,$$
$$7 + 6 = 13.$$

Substituting $-\frac{1}{2}$ for x and $6\frac{3}{4}$ for y in (1) and (2),

$$\tfrac{1}{4} + \tfrac{27}{4} = 7,$$
$$-\tfrac{1}{2} + \tfrac{27}{2} = 13.$$

EXERCISES

Solve the following systems, pair results, and check each set of roots:

1. $x + y = 5,$
   $x^2 + y^2 = 13.$

2. $x^2 - y^2 = 8,$
   $2x + y = 7.$

3. $x^2 + xy = 15,$
   $x + y = 3.$

4. $x^2 + 2xy = 21,$
   $x + 2y = 3.$

5. $x^2 + 2y = 17,$
   $2x - y = 2.$

6. $y^2 - 10x = 6,$
   $x + 3y = 13.$

7. $y^2 - 2x^2 = 7,$
   $2y + 3x = 1.$

8. $2x^2 - xy = 70,$
   $x + 4y = 23.$

9. $s^2 + 2st = 108,$
   $4s - 3t = 6.$

10. $s^2 - 3st = 22,$
    $4s + 2t = 2.$

11. $s^2 + t^2 = 169,$
    $2s + t = 22.$

12. $s^2 + 2t^2 = 27,$
    $4t - 2s = 18.$

13. $s^2 + 3ts + t^2 = 44,$
$2s - t = 0.$

14. $s^2 + ts + t^2 = 12,$
$s + t = 2.$

15. $2s^2 - ts + t^2 = 16,$
$2s - t = 5.$

16. $xy + 10 = 0,$
$4x + y = 6.$

17. $4x + y = 28,$
$2x^2 + 3xy = 98.$

18. $3x_1 + 4x_2 = 5,$
$2x_1x_2 - 6x_1 = -3.$

19. $\frac{x}{y} + \frac{y}{x} = 2,$
$3x + 2y = 5.$

20. $\frac{1}{x} - \frac{1}{y} = \frac{1}{4},$
$3x - y = 2.$

21. $\frac{x}{4} + \frac{y}{5} = 6,$
$\frac{10}{y} - \frac{10}{x} = \frac{3}{2}.$

22. $\frac{1}{x} + \frac{y}{2} = 3,$
$\frac{1}{y} + \frac{x}{2} = \frac{5}{4}.$

23. $\frac{s}{3} + \frac{t}{4} = 5,$
$\frac{6}{s} - \frac{4}{t} = \frac{2}{3}.$

24. $\frac{t+1}{s+1} = \frac{3}{2},$
$\frac{t+s^2}{s+t^2} = \frac{1}{2}.$

25. $\frac{x}{y} - \frac{y}{x} = \frac{5}{6},$
$3y + 2x = 2.$

26. $s^2 + t^2 + 2t = 40,$
$s + t + 2 = 0.$

27. $2x^2 + xy = 8,$
$y - x = \sqrt{2}.$

28. $x^2 + y^2 + 4x + 6y = 40,$
$x - 10 = y.$

29. $y + x\sqrt{15} = 0,$
$y^2 + x^3 = 16x.$

107. Homogeneous equations. If both the equations of a system are *quadratic*, an attempt to solve it by *substitution* usually gives an equation of the fourth degree. In most cases such an equation could not be solved by factoring, and at the present time its solution by any other method is beyond the student. With certain types of

systems, however, which occur more or less frequently, we can employ special devices and avoid the solution of any equation of higher degree than a quadratic. Among these systems are the so-called "homogeneous" systems.

An equation is **homogeneous** if, on being written so that one member is zero, the terms in the other member are of the same *degree* with respect to the variables.

Thus $x^2 + y^2 = xy$ and $x^2 - 3xy + y^2 = 0$ are homogeneous equations of the second degree. $2x^3 + y^3 = x^2y - 3xy^2$ is a homogeneous equation of the third degree.

The system $\left\{\begin{array}{l} xy + y^2 = x^2, \\ x^2 + 2y^2 - 3xy = 0 \end{array}\right\}$ is a homogeneous system.

Systems like $\left\{\begin{array}{l} y^2 + xy + x^2 = 7, \\ 2x^2 - y^2 + 4xy = 6 \end{array}\right\}$ are often called homogeneous systems, but strictly speaking they are not. As will be seen, the method of solving such systems is about the same as the method of solving a homogeneous system. Hence they are classed with homogeneous systems.

108. Systems having both equations quadratic. Occasionally when both equations are quadratic the terms which occur in the two are so related that the elimination of terms involving both variables or of the constant terms can be performed. The following system is of such type:

EXAMPLES

1. Solve the system $\left\{\begin{array}{ll} x^2 - xy = 2, & (1) \\ 3x^2 + 4xy = 20. & (2) \end{array}\right.$

Solution. First eliminate xy by addition (§ 47),

$(1) \cdot 4,$ $\qquad 4x^2 - 4xy = 8. \qquad (3)$

$(2) + (3),$ $\qquad 7x^2 = 28.$

Whence $\qquad x = \pm 2.$

Substituting $+2$ for x in (1), $4 - 2y = 2$, whence $y = 1$.

Substituting -2 for x in (1), $4 + 2y = 2$, whence $y = -1$.

Therefore $x = 2, -2$,

and $y = 1, -1$.

These values may be checked as usual.

The method of solving a system in which every term in each equation except the constant terms is of the second degree is as follows:

2. Solve $\begin{cases} xy + 3y^2 = 6, & (1) \\ x^2 + y^2 = 10. & (2) \end{cases}$

Solution. First we combine the two equations to obtain a homogeneous equation:

(1) · 5, $5xy + 15y^2 = 30.$ (3)

(2) · 3, $3x^2 + 3y^2 = 30.$ (4)

(3) – (4), $-3x^2 + 5xy + 12y^2 = 0.$ (5)

Solving (5) for x in terms of y, $x = 3y$, (6)

and $x = -\frac{4y}{3}$. (7)

Substituting $3y$ for x in (2), $9y^2 + y^2 = 10.$ (8)

From (8), $y = \pm 1.$ (9)

Substituting from (9) in (6), $x = \pm 3.$

Substituting $-\frac{4y}{3}$ for x in (2), $\frac{16y^2}{9} + y^2 = 10.$ (10)

From (10), $y = \pm \frac{3}{5}\sqrt{10}.$ (11)

Substituting from (11) in (7), $x = \mp \frac{4}{5}\sqrt{10}.$

When $x =$	3	-3	$+\frac{4}{5}\sqrt{10}$	$-\frac{4}{5}\sqrt{10}$
then $y =$	1	-1	$-\frac{3}{5}\sqrt{10}$	$+\frac{3}{5}\sqrt{10}$

Each pair of values can be checked as usual.

A quadratic system in which one equation is homogeneous is easier to solve than the system of Example 2, as can be seen from what follows:

3. Solve the system $\begin{cases} 3x^2 + 2y^2 = 7xy, & (1) \\ x^2 + y^2 - 5x = 3. & (2) \end{cases}$

HINTS. Solving (1) for x, $\quad x = 2y, \qquad (3)$

and $\quad x = \frac{y}{3}. \qquad (4)$

We may now substitute from (3) in (2) and from (4) in (2), solve the resulting equations, and then complete the solution as in Example 2.

EXERCISES

Solve, pair results, and check each set of real roots:

1. $x^2 + y^2 = 20,$
$3y^2 + x^2 = 28.$

2. $3x^2 + 2xy = 7,$
$2x^2 - 3xy = -4.$

3. $3y^2 - xy = 2,$
$2y^2 + 3xy = 38.$

4. $y^2 - xy = 2x^2,$
$2x^2 + xy = 16.$

5. $x^2 - 3y + 3 = 0,$
$y^2 + x^2 = 25.$

6. Solve Example 3 completely.

7. $x^2 + 2xy - y^2 = 32,$
$2x^2 - 3xy + y^2 = 0.$

8. $2x^2 - xy + 2y^2 = 12,$
$2x^2 + xy + 2y^2 = 8.$

9. $2x^2 + y^2 = 3xy,$
$x^2 + 3xy = 16.$

10. $s^2 - 3st = 4,$
$t^2 + s^2 = 4.$

11. $s^2 + 2st + 4t^2 = 13,$
$t^2 + 8 = -2st.$

109. Symmetric systems. A system of equations in x and y is **symmetric** if the system is not altered by substituting x for y and y for x.

Thus $x^2 + y^2 = 6$ and $x + y = 11$ is a symmetric system, but $x^2 - y^2 = 6$ and $x + y = 11$ is not.

Certain symmetric systems or systems which are nearly symmetric can be easily solved by the method of substitution. Of such the following are types:

$$A\begin{cases} x \pm y = 6, \\ xy = 4. \end{cases} \quad B\begin{cases} x^2 + y^2 = 25, \\ x \pm y = 2. \end{cases} \quad C\begin{cases} 4x^2 + y^2 = 13, \\ 2x \pm y = 11. \end{cases}$$

A few other systems which are symmetric or nearly so are more easily solved by certain special methods. The following list contains typical systems, and the methods applicable are given in Exercises 1, 10, and 12.

EXERCISES

1. Solve $\begin{cases} x^2 + y^2 = 37, & (1) \\ xy = 6. & (2) \end{cases}$

HINTS. These equations can be combined in such a way as to obtain definite values for $x + y$ and $x - y$ as follows:

(2) · 2,	$2xy = 12.$	(3)
(1) + (3),	$x^2 + 2xy + y^2 = 49.$	(4)
From (4),	$x + y = \pm 7.$	(5)
(1) − (3),	$x^2 - 2xy + y^2 = 25.$	(6)
From (6),	$x - y = \pm 5.$	(7)

(5) and (7) combined give four systems of equations:

$$A\begin{cases} x + y = 7, & (8) \\ x - y = 5. & (9) \end{cases} \qquad C\begin{cases} x + y = -7, & (11) \\ x - y = 5. & (9) \end{cases}$$

$$B\begin{cases} x + y = 7, & (8) \\ x - y = -5. & (10) \end{cases} \qquad D\begin{cases} x + y = -7, & (11) \\ x - y = -5. & (10) \end{cases}$$

The solution of systems A, B, C, and D is left to the student.

The pairs of roots found for the four systems A, B, C, and D will check in the original system.

2. $\begin{matrix} x^2 + y^2 = 100, \\ xy = 48. \end{matrix}$ 3. $\begin{matrix} x^2 + xy + y^2 = 19, \\ xy = 6. \end{matrix}$ 4. $\begin{matrix} x^2 + 4y^2 = 20, \\ xy = 4. \end{matrix}$

5. $9x^2 + y^2 = 61,$
$xy = 10.$

6. $4x^2 + xy + y^2 = 85,$
$xy = 12.$

7. $4x^2 + 25y^2 = 41,$
$xy = 2.$

8. $x^2 - xy + y^2 = 8,$
$xy + 1 = 0.$

9. $\frac{x^2}{3} - xy + \frac{4y^2}{3} = \frac{7}{3},$
$x = \frac{6}{y}.$

10. $\frac{1}{x^2} + \frac{1}{y^2} = 13,$
$\frac{1}{xy} = 6.$

HINT. To clear of fractions in Exercise 10 would merely increase the difficulty of solution. Instead we solve in a manner similar to that of Exercise 1 (see Exercise 6, p. 78).

$$\frac{2}{xy} = 12.$$

Then $$\frac{1}{x^2} + \frac{2}{xy} + \frac{1}{y^2} = 25.$$

Whence $$\frac{1}{x} + \frac{1}{y} = \pm 5.$$

In like manner $$\frac{1}{x} - \frac{1}{y} = \pm 1.$$

11. $\frac{1}{x^2} + \frac{1}{y^2} = 25,$
$\frac{1}{xy} = 12.$

12. $\frac{1}{x^2} + \frac{1}{y^2} = \frac{5}{4},$
$\frac{1}{x} - \frac{1}{y} = \frac{1}{2}.$

HINTS. $\frac{1}{x^2} - \frac{2}{xy} + \frac{1}{y^2} = \frac{1}{4}.$

Then $\frac{2}{xy} = 1$, etc.

13. $\frac{1}{3x} + \frac{1}{2y} = \frac{13}{6},$
$\frac{1}{9x^2} + \frac{1}{4y^2} = \frac{97}{36}.$

14. $\frac{1}{x^2} - \frac{1}{xy} + \frac{1}{y^2} = 7,$
$\frac{1}{xy} = 6.$

15. $\frac{1}{x^2} - \frac{1}{xy} + \frac{1}{y^2} = 13,$
$\frac{1}{x} + \frac{1}{y} = 7.$

110. Equivalent systems. **Equivalent systems** of equations are systems which have the same set or sets of roots.

If the two systems

$$A\begin{cases} x^2 - y^2 = 12, & (1) \\ x + y = 6. & (2) \end{cases} \quad \text{and} \quad B\begin{cases} x - y = 2, & (3) \\ x + y = 6. & (4) \end{cases}$$

are solved, only one pair of roots, $x = 4$ and $y = 2$, is obtained for A and the same pair for B. Systems A and B are equivalent, though usually a system which consists of a linear and a quadratic has two sets of roots and hence cannot be equivalent to a linear system (see page 183, second paragraph).

Sometimes an equation simpler than either of those given in a system can be derived from this system by dividing the left and right members of the first equation by the corresponding members of the second. In systems like those of the following list the equation so obtained taken with one of the first two gives an equivalent system more easily solved than the original one.

EXERCISES

Solve, using division where possible, pair results, and check each set of real roots:

1. $\begin{aligned} x^2 - 4y^2 &= 20, \\ x + 2y &= 2. \end{aligned}$

HINT. Division gives the equivalent system

$$\begin{aligned} x - 2y &= 10, \\ x + 2y &= 2. \end{aligned}$$

2. $\begin{aligned} x^2 - y^2 &= 77, \\ x - y &= 7. \end{aligned}$

3. $\begin{aligned} x^2 - 2xy &= 187, \\ 2y - x &= -17. \end{aligned}$

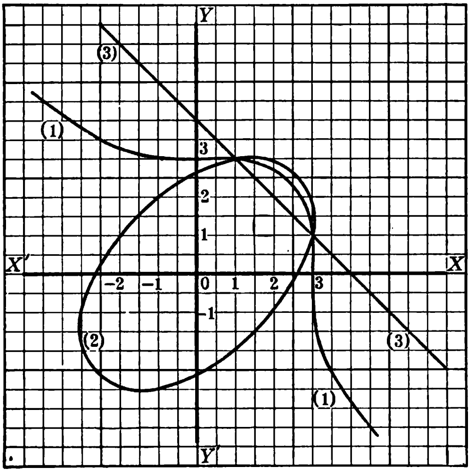
4. $\begin{aligned} 4x^2 - y^2 &= 16, \\ 2x + y &= 8. \end{aligned}$

5. $\begin{aligned} R^2h - 75 &= 0, \\ Rh &= 15. \end{aligned}$

6. $\begin{aligned} \frac{1}{x^2} - \frac{1}{y^2} &= 15, \\ \frac{1}{x} + \frac{1}{y} &= 3. \end{aligned}$

7. $\begin{aligned} x^3 + y^3 &= 28, \\ x + y &= 4. \end{aligned}$

In the following figure, (1), (2), and (3) are the graphs of $x^3 + y^3 = 28$, $x^2 - xy + y^2 = 7$, and $x + y = 4$ respectively. These equations are all used in the solution of Exercise 7. The graph makes clear in a striking way that the system (1) and (3) is equivalent to the system (2) and (3).



8. $$\begin{aligned} x^3 - 8y^3 &= 35, \\ x - 2y &= 5. \end{aligned}$$

9. $$\begin{aligned} \frac{1}{x} + \frac{1}{y} &= 1, \\ \frac{1}{y^2} - \frac{1}{x^2} &= 5. \end{aligned}$$

10. $$\begin{aligned} \frac{1}{x^3} + \frac{1}{y^3} &= 152, \\ \frac{1}{x} + \frac{1}{y} &= 8. \end{aligned}$$

MISCELLANEOUS EXERCISES

Solve by any method and pair results. If any system cannot be solved algebraically by the methods previously given, solve it graphically.

1. $$\begin{aligned} y^2 + xy &= 8, \\ x + y &= 4. \end{aligned}$$

2. $$\begin{aligned} x^2 - 9y^2 &= 7, \\ x - 3y &= 1. \end{aligned}$$

3. $$\begin{aligned} x^2 - x - 2y &= 0, \\ y^2 - 4y + x &= 0. \end{aligned}$$

4. $$\begin{aligned} x^2 + y^2 &= 25, \\ x + y &= 7. \end{aligned}$$

5. $\frac{x}{y} + 3 = 0,$
$xy + 12 = 0.$

6. $(x + y)^2 = 9,$
$(x - y)^2 = 49.$

7. $2x^2 + y^2 = 33,$
$x^2 + 2y^2 = 54.$

8. $3h^2 - 8k^2 = 40,$
$5h^2 + k^2 = 81.$

9. $4R_1^2 + 3 = 9R_2^2,$
$12R_1^2 + R_2^2 = \frac{31}{9}.$

10. $xy + x = 18,$
$xy + y = 20.$

11. $x^2 + y^2 = 169,$
$xy = 60.$

12. $x^2 = y,$
$xy = 8.$

13. $x - xy = 5,$
$2y + xy = 6.$

14. $x^3 - y^3 = 19,$
$x^2 + xy + y^2 = 19.$

15. $4n^2 + 7m^2 = 9,$
$2n^2 - \frac{9}{2} = m^2.$

16. $5W_1^2 - 6.8W_2^2 = 99.55,$
$W_1^2 - W_2^2 = 20.$

17. $xy + 2y^2 = 2,$
$3xy + 5y^2 = 2.$

18. $x^2 + 2xy + 2y^2 = 10,$
$3x^2 - xy - y^2 = 51.$

19. $y^2 + x = 7,$
$x^2 + y = 11.$

20. $x^2 + xy + y^2 = 7,$
$x^2 + y^2 = 10.$

21. $x^2 + xy + y = 0,$
$x^2 + xy + x = 0.$

22. $\frac{1}{x^2} + \frac{1}{y^2} = 13,$
$\frac{1}{x} - \frac{1}{y} = 1.$

23. $\frac{1}{x^3} - \frac{1}{y^3} = 7,$
$\frac{1}{x} - \frac{1}{y} = 1.$

24. $4x^2 + y^2 = 289,$
$xy = 60.$

25. $x^{\frac{1}{2}} + y^{\frac{1}{2}} = 9,$
$x^{\frac{1}{2}}y^{\frac{1}{2}} = 20.$

26. $x^2 + y^2 + x + y = 58,$
$x + y = 5.$

27. $9x \div y = 18 = xy.$

28. $4x + \frac{1}{y} = 46 = \frac{26x}{5} - \frac{1}{y}.$

29. $\frac{1}{x-2} + \frac{1}{y-2} = \frac{3}{4},$
$\frac{1}{x} - \frac{1}{y} = \frac{1}{12}.$

30. $3xy = x^2y^2 - 88,$
$x - y = 6.$

31. $xy = c,$
$x + y = a.$

32. $x^{-2} - y^{-2} = 6,$
$x^{-1} + y^{-1} = 2.$

33. $x - y = 16,$
$x^{\frac{1}{2}} - y^{\frac{1}{2}} = 2.$

34. $\dfrac{x-1}{y-1} = 3,$
$\dfrac{y^2 + y + 1}{x^2 - x + 1} = \dfrac{13}{43}.$

35. $4x^2 - 13xy + 9y^2 = 9,$
$xy - y^2 = 3.$

36. $x^2 + 2xy = 16,$
$3x^2 - 4xy + 2y^2 = 6.$

37. $x^3 = y^3 + 37,$
$x^2y = xy^2 + 12.$

PROBLEMS

(Reject all results which do not satisfy the conditions of the problems.)

1. Find two numbers whose difference is 6 and the difference of whose squares is 120.

2. The sum of two numbers is 20 and the sum of their squares is 202. Find the numbers.

3. Find two numbers whose sum plus their product is 132 and whose quotient is 3.

4. It takes 56 rods of fence to inclose a rectangular lot whose area is one acre. Find the dimensions of the lot.

5. The area of a right triangle is 180 square feet and its hypotenuse is 41 feet. Find the legs.

6. The area of a pasture containing 15 acres is doubled by increasing its length and its breadth by 20 rods. What were the dimensions at first?

7. The difference of the areas of two squares is 495 square feet and the difference of their perimeters is 60 feet. Find a side of each square.

8. The area of a rectangular field is $43\frac{1}{5}$ acres and one diagonal is 120 rods. Find the perimeter of the field.

9. The value of a certain fraction is $\frac{2}{3}$. If the fraction is squared and 44 is subtracted from both the numerator and the denominator of this result, the value of the fraction thus formed is $\frac{5}{14}$. Find the original fraction.

10. Two men together can do a piece of work in $4\frac{4}{5}$ days. One man requires 4 days less than the other to do the work alone. Find the number of days each requires alone.

11. The perimeter of a rectangle is 250 feet and its area is 214 square yards. Find the length and the width.

12. The base of a triangle is 8 inches longer than its altitude and the area is $1\frac{2}{3}$ square feet. Find the base and altitude of the triangle.

13. The volumes of two cubes differ by 316 cubic inches and their edges differ by 4 inches. Find the edge of each.

14. The perimeter of a right triangle is 80 feet and its area is 240 square feet. Find the legs and the hypotenuse.

15. The perimeter of a rectangle is $7a$ and its area is a^2. Find its dimensions.

16. A man travels from A to B, 30 miles, by boat and from B to C, 120 miles, by rail. The trip required 6 hours. He returned from C to B by a train running 10 miles per hour faster, and from B to A by the same boat. The return trip took 5 hours. Find the rate of the boat and of each train.

17. There were 1400 fewer reserved seats at a certain sale than of unreserved seats, and the price of the latter was 15 cents less than the price of the former. The total proceeds were $490, of which $250 came from the reserved seats. Find the number of each kind of seats and the price of each.

18. If a two-digit number be multiplied by the sum of its digits, the product is 324; and if three times the sum of its digits be added to the number, the result is expressed by the digits in reverse order. Find the number.

19. The sum of the radii of two circles is 31 inches and the difference of their areas is 155π square inches. Find the radii.

20. The yearly interest on a certain sum of money is \$42. If the sum were \$200 more and the interest 1% less, the annual income would be \$6 more. Find the principal and the rate.

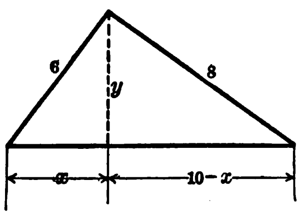
21. A wheelman leaves A and travels north. At the same time a second wheelman leaves a point 3 miles east of A and travels east. One and one-third hours after starting, the shortest distance between them is 17 miles, and $3\frac{1}{3}$ hours later the distance is 53 miles. Find the rate of each.

22. A starts out from P to Q at the same time B leaves Q for P. When they meet, A has gone 40 miles more than B. A then finishes the journey to Q in 2 hours and B the journey to P in 8 hours. Find the rates of A and B and the distance from P to Q.

23. A leaves P going to Q at the same time that B leaves Q on his way to P. From the time the two meet, it requires $6\frac{2}{3}$ hours for A to reach Q, and 15 hours for B to reach P. Find the rate of each, if the distance from P to Q is 300 miles.

GEOMETRICAL PROBLEMS

1. The sides of a triangle are 6, 8, and 10. Find the altitude on the side 10.



HINT. From the accompanying figure we easily obtain the system

$$\begin{cases} x^2 + y^2 = 36, \\ (10 - x)^2 + y^2 = 64. \end{cases}$$

2. The sides of a triangle are 8, 15, and 17. Find the altitude of the triangle on the side 17 and the area of the triangle.

3. The sides of a triangle are 11, 13, and 20. Find the altitude on the side 20 and the area of the triangle.

4. The sides of a triangle are 13, 14, and 15. Find the altitude on the side 14 and the area of the triangle.

5. The sides of a triangle are 12, 17, and 25. Find the altitude on the side 12 and the area of the triangle.

6. Find correct to two decimals the altitude on the side 16 of a triangle whose sides are 16, 20, and 24 respectively.

7. The parallel sides of a trapezoid are 14 and 26 respectively, and the two nonparallel sides are 10 each. Find the altitude of the trapezoid.

HINT. Let $ABCD$ be a trapezoid. Draw CE parallel to DA and CF perpendicular to AB.

Then

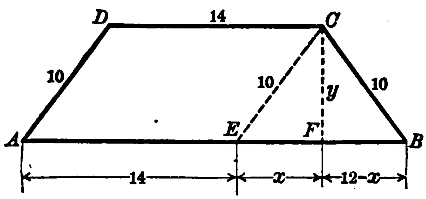
$$EC = 10,$$

and

$$AE = 14,$$

and

$$EB = 26 - 14, \text{ or } 12.$$



If we let $EF = x$, FB must equal $12 - x$; then we can obtain the system of equations

$$\begin{cases} x^2 + y^2 = 100, \\ (12 - x)^2 + y^2 = 100. \end{cases}$$

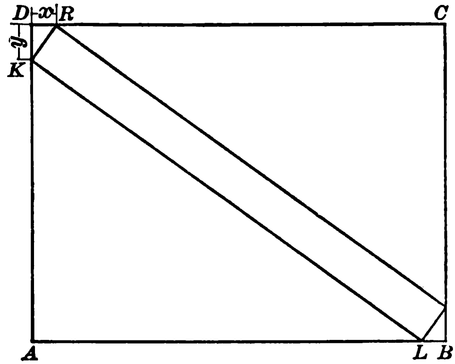
8. The two nonparallel sides of a trapezoid are 10 and 17 respectively, and the two bases are 9 and 30 respectively. Find the altitude of the trapezoid.

9. The bases of a trapezoid are 15 and 20 respectively, and the two nonparallel sides are 29 and 30. Find the altitude of the trapezoid and the area.

10. The sides of a trapezoid are 7, 9, 20, and 24. The sides 24 and 9 are the bases. Find the altitude and the area.

11. The sides of a trapezoid are 21, 27, 40, and 30. The sides 21 and 40 are parallel. Find the altitude and the area of the trapezoid.

12. The sides of a trapezoid are 23, 85, 100, and x. The sides 23 and 100 are the bases, and each is perpendicular to the side x. Find x and the area of the trapezoid.



13. The area of a triangle is 1 square foot. The altitude on the first side is 16 inches. The second side is 14 inches longer than the third. Find the three sides.

14. A rectangular tank is 8 feet 6 inches long and 6 feet 8 inches wide. A board 10 inches wide is laid diagonally on the floor. What two equations must be solved to determine the length of the longest board that can be thus laid?

Hints. Let $DR = x$ and $DK = y$. The triangle DKR is similar to the triangle AKL.

CHAPTER XVI

PROGRESSIONS

111. A sequence of numbers. In all fields of mathematics we frequently encounter groups of three or more numbers, selected according to some law and arranged in a definite order, whose relations to each other and to other numbers we wish to study.

There is an unlimited variety of such groups, or successions, of numbers. Only two simple types will be considered here.

112. Arithmetical progression. An **arithmetical progression** is a succession of terms in which each term after the first is formed by adding the same number to the preceding one.

Thus, if a denotes the first term and d the common number added, any arithmetical progression is represented by

$$a,\ a + d,\ a + 2d,\ a + 3d,\ a + 4d, \cdots.$$

This common number d is called the **common difference** and may be any number, *positive* or *negative*. It may be found for any given arithmetical progression by subtracting any term from the term which follows it.

The numbers 3, 7, 11, 15, $\cdots$ form an arithmetical progression, since any term, after the first, minus the preceding one gives 4. Similarly, 12, 6, 0, -6, $-12, \cdots$ is an arithmetical progression, since any term, after the first, minus the preceding one gives the common difference -6. In like manner $\frac{7}{2}$, 5, $6\frac{1}{2}, \cdots$ is an arithmetical progression whose common difference is $1\frac{1}{2}$.

ORAL EXERCISES

State the first four terms of the arithmetical progression if:

1. $a = 2$, $d = 3$.
2. $a = 5$, $d = 4$.
3. $a = 10$, $d = 6$.
4. $a = 20$, $d = 5$.
5. $a = 18$, $d = -3$.
6. $a = 100$, $d = -10$.
7. $a = 20x$, $d = -2x$.
8. $a = x$, $d = 2x$.
9. $a = 17m$, $d = -2m$.
10. $a = \sqrt{5}$, $d = 1 + \sqrt{5}$.

From the following select the arithmetical progressions, and in each arithmetical progression find the common difference:

11. $1, 10, 19, \cdots$.
12. $4, 12, 36, \cdots$.
13. $19, 11, 3, \cdots$.
14. $9, 12, \frac{1}{2}, 16, \cdots$.
15. $3, \frac{20}{3}, \frac{31}{3}, \cdots$.
16. $8, 9\frac{2}{5}, 10\frac{4}{5}, \cdots$.
17. $15, 3, -12, \cdots$.
18. $6a, 10a, 14a, \cdots$.
19. $18a, 14a, 12a, \cdots$.
20. $5a, 5a + 3, 5a + 6, \cdots$.
21. $3x, x - 2, -x - 4, \cdots$.
22. $2x - 1, x, 1, 2 - x, \cdots$.
23. $-5\sqrt{a}, -2\sqrt{a}, \sqrt{a}, 4\sqrt{a}, \cdots$.
24. $5\sqrt{x} - 1, 4\sqrt{x} - 2, 3\sqrt{x} - 3, \cdots$.

113. The last or *n*th term of an arithmetical progression. In the arithmetical progression

$$a,\ a + d,\ a + 2d,\ a + 3d,\ a + 4d,\ \cdots$$

one observes that the coefficient of d in each term is 1 less than the number of the term. Hence the nth, or general, term is $a + (n - 1)d$. If l denotes the nth term, we have

$$l = a + (n - 1)d.$$

EXERCISES

Find the required terms of the following arithmetical progressions:

1. The fifteenth term of 2, 7, 12, $\cdots$.

Solution. Here $a = 2$, $d = 5$, and $n = 15$.

Hence $$l = a + (n-1)d$$

becomes $$l = 2 + 14 \cdot 5 = 72.$$

2. The eighth term of 1, 4, 7, $\cdots$.

3. The eleventh term of 15, 9, 3, -3, $\cdots$.

4. The twenty-first term of a, $4a$, $7a$, $\cdots$.

5. The sixteenth term of $9x$, x, $-7x$, $\cdots$.

6. The sixth term of $\frac{10}{3}$, $\frac{6}{3}$, $\frac{2}{3}$, $\cdots$.

7. The eleventh term of $-\frac{1}{7}$, $\frac{3}{7}$, 1, $\cdots$.

8. The sixth term of $\sqrt{3}$, $4\sqrt{3}$, $7\sqrt{3}$, $\cdots$.

9. The eighth term of $3\sqrt{5}$, $\sqrt{5}$, $-\sqrt{5}$, $-3\sqrt{5}$, $\cdots$.

10. The eighth term of $5 + a$, $7 + 3a$, $9 + 5a$, $\cdots$.

11. The twelfth and the twentieth terms of 0, $-5x + 2$, $4 - 10x$, $\cdots$.

12. The tenth term of $\sqrt{a} + 3$, $5\sqrt{a} + 7$, $9\sqrt{a} + 11$, $\cdots$.

13. The tenth and the twentieth terms of $9\sqrt{a} - 3$, $5\sqrt{a}$, $\sqrt{a} + 3$, $\cdots$.

14. The fourteenth term of $\frac{\sqrt{x}}{2}$, $\frac{5\sqrt{x}}{2}$, $\frac{9\sqrt{x}}{2}$, $\cdots$.

15. The ninth and the twelfth terms of $\sqrt{3}$, $\frac{6}{\sqrt{3}}$, $3\sqrt{3}$, $\cdots$.

16. The eighth and the sixteenth terms of $\frac{\sqrt{x}}{2} + 1$, $\frac{3\sqrt{x}}{2} + 2$, $\frac{5\sqrt{x}}{2} + 3$, $\cdots$.

17. The fifteenth term of $\frac{2\sqrt{a}}{3} - 12$, -5, $\frac{2(3 - \sqrt{a})}{3}$, $\cdots$.

18. The twenty-ninth term of $a, a+d, a+2d, \cdots$.

19. The mth term of $a, a+d, a+2d, \cdots$.

20. The $(m-1)$th term of $a, a+d, a+2d, \cdots$.

21. The $(n-2)$th term of $a, a+5, a+10, \cdots$.

22. The $(n-5)$th term of $\frac{3}{\sqrt{3}}, 3\sqrt{3}, 5\sqrt{3}, \cdots$.

23. Find the $(n-3)$th term of $\sqrt{5}-1, 2\sqrt{5}-2, 3(\sqrt{5}-1), \cdots$.

24. Find the nth term of $\frac{1}{a}, \frac{a-1}{a}, 2-\frac{3}{a}, \cdots$.

25. The first and third terms of an arithmetical progression are 2 and 22. Find the seventh term; the nth term.

26. The first and second terms of an arithmetical progression are r and s respectively. Find the third term and the nth term.

27. The edges of a box are consecutive even integers with n the least. Express in terms of n, (a) the sum of the edges; (b) the area of the faces; (c) the volume.

28. A body falls 16 feet the first second, 48 feet the next, 80 feet the next, and so on. How far does it fall during the twelfth second? during the nth second?

29. The digits of a certain three-digit number are in arithmetical progression. If their sum is 24 and the sum of their squares is 194, find the number.

114. Arithmetical means. The **arithmetical means** between two numbers are numbers which form, with the two given ones as the first and the last term, an arithmetical progression.

The insertion of one or more arithmetical means between two numbers is performed as in the following examples:

EXAMPLES

1. Insert four arithmetical means between 7 and 72.

Solution. $l = a + (n-1)d$.

There will be six terms in all.

Therefore $\qquad 72 = 7 + (6-1)d.$

Solving, $\qquad d = 13.$

The required arithmetical progression is 7, 20, 33, 46, 59, 72.

2. Insert one arithmetical mean between h and k.

Solution. $l = a + (n-1)d$.

There will be three terms in all.

Therefore $\qquad k = h + 2d.$

Solving, $\qquad d = \dfrac{k-h}{2}.$

Therefore the progression is h, $h + \dfrac{k-h}{2}$, k, or h, $\dfrac{h+k}{2}$, k.

It follows from the above that the arithmetical mean between two numbers is one half of their sum.

$+ (3-1)d$

$15 = 3 + 2d$

EXERCISES

Insert one arithmetical mean between:

1. 3 and 15. **2.** 3 and 27. **3.** h and $5h$. **4.** m and n.

Insert two arithmetical means between:

5. 2 and 8. **6.** 5 and 29. **7.** 49 and 217. **8.** x and $x + 6y$. **9.** x and $3y$. **10.** h and k.

Insert three arithmetical means between:

11. 10 and 34. **12.** -5 and 31. **13.** $-12x$ and $44x$. **14.** h and k.

15. Insert seven arithmetical means between -13 and 131.

16. Insert five arithmetical means between $-\frac{8}{3}$ and $\frac{46}{3}$.

17. Insert four arithmetical means between $-2\sqrt{5}$ and $18\sqrt{5}$.

18. Insert five arithmetical means between $7a-3b$ and $13a+9b$.

19. Insert six arithmetical means between $\sqrt{3}$ and $\dfrac{27}{2\sqrt{3}}$.

20. Insert two arithmetical means between $\dfrac{3}{1-\sqrt{3}}$ and $4\sqrt{3}-6$.

21. In going a distance of 1 mile an engine increased its speed uniformly from 15 miles per hour to 25 miles per hour. What was the average velocity in miles per hour during that time? How long did it require to run the mile?

22. At the beginning of the third second the velocity of a falling body is 64 feet per second, and at the end of the third second it is 96 feet per second. What is its average velocity in feet per second during the third second? How many feet does it fall during the third second?

23. The velocity of a body falling from rest increases uniformly and is 32 feet per second at the end of the first second. What is the average velocity in feet per second during the first second? How many feet does the body fall during the first second? the second second?

24. Find the average length of twenty lines whose lengths in inches are the first twenty odd numbers.

25. Find the average length of fifteen lines whose lengths in inches are given by the consecutive even numbers beginning with 58.

26. With the conditions of Problem 23 determine the average velocity per second of a body which has fallen for 12 seconds.

27. A certain distance is separated into ten intervals, the lengths of which are in arithmetical progression. If the shortest interval is 1 inch and the longest 37 inches, find the others.

115. Sum of a series. The indicated sum of several terms of an arithmetical progression is called an arithmetical series. The formula for the sum of n terms of an **arithmetical series** may be obtained as follows:

$$S=a+(a+d)+(a+2d)+\cdots+(l-2d)+(l-d)+l. \quad (1)$$

Reversing the order of the terms in the second member of (1),

$$S=l+(l-d)+(l-2d)+\cdots+(a+2d)+(a+d)+a. \quad (2)$$

Adding (1) and (2),

$$2S=(a+l)+(a+l)+(a+l)+\cdots+(a+l)+(a+l)+(a+l)=n(a+l).$$

Therefore $$S=\frac{n}{2}(a+l).$$

EXAMPLE

Required the sum of the integers from 7 to 92, inclusive.

Solution. $n=86$, $a=7$, $l=92$.

Substituting in $S=\frac{n}{2}(a+l)$,

$$S=\frac{86(7+92)}{2}=4257.$$

Therefore the sum of the integers from 7 to 92 is 4257.

ORAL EXERCISES

Using the formula $S=\frac{n}{2}(a+l)$, find the sum of the following:

1. The six-term series in which $a=2$ and $l=17$.
2. The ten-term series in which $a=1$ and $l=46$.
3. The twelve-term series in which $a=-12$ and $l=21$.
4. The seven-term series in which $a=3$ and $l=63$.
5. The twenty-term series in which $a=5$ and $l=85$.
6. The twelve-term series in which $a=-175$ and $l=125$.

The formula $S = \frac{n}{2}(a + l)$ enables one to find the sum of a series if the first term, the last term, and the number of the terms are given. If, however, the last term is not given, but instead the common difference is given or evident, we can use a formula obtained by substituting in $S = \frac{n}{2}(a + l)$ the value of l from the formula $l = a + (n - 1)d$, on page 205.

Then $$S = \frac{n}{2}[a + a + (n - 1)d],$$

or $$\mathbf{S = \frac{n}{2}[2a + (n - 1)d].}$$

EXAMPLE

Required the sum of the first fifty-nine terms of the progression 2, 9, 16, $\cdots$.

Solution. $n = 59$, $a = 2$, and $d = 7$.

Substituting in $S = \frac{n}{2}[2a + (n - 1)d]$,

$$S = \frac{59}{2}(4 + 58 \cdot 7) = \frac{59}{2}(410) = 12{,}095.$$

EXERCISES

Find the sum of the following:

1. The first eight terms of the series in which $a = 2$ and $d = 4$.

2. The first ten terms of the series in which $a = 8$ and $d = 5$.

3. The first nine terms of the series in which $a = 1$ and $d = 11$.

4. The first fifteen terms of the series in which $a = 17$ and $d = 3$.

5. The first twenty terms of the series in which $a = 100$ and $d = -5$.

6. The first eight terms of the series $2 + 4 + 6 + \cdots$.

7. The first ten terms of the series $1 + 9 + 17 + \cdots$.

8. The first ten terms of the series $-8 + (-4) + 0 + \cdots$.

9. The first eighteen terms of the series $1 + 5 + 9 + \cdots$.

10. The first twenty terms of the series $10 + 8 + 6 + \cdots$.

11. The first twelve terms of $1 + \frac{3}{2} + 2, \cdots$.

12. The first twelve terms of $15 + 12\frac{1}{2} + 10 + \cdots$.

13. The first one hundred integers.

14. The first one hundred even numbers.

15. The first one hundred odd numbers.

16. Show that the sum of the first n even numbers is $n(n + 1)$.

17. Show that the sum of the first n odd numbers is n^2.

18. Find the sum of the even numbers between 247 and 539.

19. How many of the positive integers beginning with 1 are required to make their sum 861?

HINT. Substitute in the formula $S = \frac{n}{2}[2a + (n - 1)d]$ and solve for n.

20. How many terms must constitute the series $7 + 10 + 13 + \cdots$ in order that its sum may be 242?

21. Beginning with 90 in the progression 78, 80, 82, $\cdots$, how many terms are required to give a sum of 372?

22. The second term of an arithmetical progression is -2 and the eighth term is 22. Find the eleventh term.

23. Find the sum of the first t terms of $\frac{1}{a}, \frac{a-1}{a}, \frac{2a-3}{a}, \cdots$.

24. If $l = 25$, $a = 1$, and $d = 4$, find n and s.

25. If $a = -20$, $d = 11$, and $s = 216$, find n and l.

26. If $d = -9$, $n = 15$, and $s = 0$, find a and l.

27. The first and second terms of an arithmetical progression are h and k respectively. Find the sum of n terms.

28. If $s = 9h$, $a = 12 - 10h$, and $n = 9$, find l and d.

29. If $s = 66\sqrt{3}$, $a = -4\sqrt{3}$, and $d = 2\sqrt{3}$, find n and l.

30. A clock strikes the hours but not the half hours. How many times does it strike in a day?

31. A car running 15 miles an hour is started up an incline, which decreases its velocity $\frac{1}{2}$ of a foot per second. (*a*) In how many seconds will it stop? (*b*) How far will it go up the incline?

32. A car starts down a grade and moves 3 inches the first second, 11 inches the second second, 19 inches the third second, and so on. (*a*) How fast does it move in feet per second at the end of the thirtieth second? (*b*) How far has it moved in the thirty seconds?

33. An elastic ball falls from a height of 24 inches. On each rebound it comes to a point $\frac{1}{2}$ inch below the height reached the time before. How often will it drop before coming to rest? Find the total distance through which it has moved.

34. The digits of a three-digit number are in arithmetical progression. The first digit is 3 and the number is $20\frac{1}{2}$ times the sum of its digits. Find the number.

35. A clerk received \$60 a month for the first year and a yearly increase of \$75 for the next nine years. Find his salary for the tenth year and the total amount received.

36. If a man saves \$200 a year and at the end of each year places this sum at simple interest at 6%, what will be the amount of his savings at the time of the sixth annual deposit?

37. Assuming that a ball is not retarded by the air, determine the number of seconds it will take to reach the ground if

dropped from the top of the Washington Monument, which is 555 feet high. With what velocity will it strike the ground?

HINT. See Exercise 28, p. 207.

38. A ball thrown vertically upward rose to a height of 256 feet. In how many seconds did it begin to fall? With what velocity was it thrown?

39. By Exercise 28, p. 207, it is seen that a falling body obeys the law of an arithmetical progression. Show from the data of that exercise that the general formula $S = \frac{n}{2}(2a + (n-1)d)$ becomes the special one $S = \frac{gt^2}{2}$, which is used in physics for all such problems.

40. A ball thrown vertically upward returned to the ground 6 seconds later. How high did it rise? With what velocity was it thrown?

41. A and B start from the same place at the same time and travel in the same direction. A travels 20 miles per hour. B goes 30 miles the first hour, 26 miles the second, 22 miles the third, and so on. When are they together?

NOTE. In the earliest mathematical work known a problem is found which involves the idea of an arithmetical progression. In the papyrus of the Egyptian priest Ahmes, who lived nearly two thousand years before Christ, we read in essence, "Divide 40 loaves among 5 persons so that the numbers of loaves that they receive shall form an arithmetical progression, and so that the two who receive the least bread shall together have one seventh as much as the others." From that time to this, the subject has been a favorite one with mathematical writers, and has been extended so widely that it would require many volumes to record all of the discoveries regarding the various kinds of series.

116. Geometrical progression. A **geometrical progression** is a succession of terms in which each term after the first is formed by multiplying the preceding one always by the same number.

Thus, if a denotes the first term and r the common multiplier, then any geometrical progression is represented by $a, ar, ar^2, ar^3, ar^4, \cdots$.

The common multiplier is called the *ratio*. It is evident from the above that the ratio r in a geometrical progression is found by dividing any term by the preceding one.

The numbers 2, 10, 50, 250, $\cdots$ form a geometrical progression, since any term, after the first, divided by the preceding one gives the same number 5. Similarly, the numbers $3, -3\sqrt{2}, 6, -6\sqrt{2}, \cdots$ form a geometrical progression, since any term, after the first, divided by the preceding one gives the common ratio $-\sqrt{2}$.

ORAL EXERCISES

Determine which of the following are geometrical progressions and state the ratio in each case:

1. $1, 3, 9, 27, \cdots$.

2. $2, 4, 16, \cdots$.

3. $2, 6, 18, \cdots$.

4. $5, 1, \frac{1}{5}, \cdots$.

5. $18, -3, \frac{1}{2}, \cdots$.

6. $2, \frac{1}{2}\sqrt{2}, \frac{1}{4}\sqrt{2}, \cdots$.

7. $\frac{1}{3}\sqrt{6}, \sqrt{6}, \sqrt{54}, \cdots$.

8. $\sqrt{\frac{2}{4}}, -\sqrt{\frac{2}{2}}, 2, \cdots$.

9. $7a, 35a, 175a, \cdots$.

10. $8\sqrt{5}, -2\sqrt{5}, \sqrt{5}, \cdots$.

11. $5x^2, 10x^2, 20x^2, \cdots$.

12. $3y^{\frac{1}{2}}, 12y, 48y^{\frac{3}{2}}, \cdots$.

13. Find the condition under which a, b, and c form a geometrical progression.

State in order the first four terms of the geometrical progression in which the first term is

14. 1 and the ratio is 4.

15. 3 and the ratio is 10.

16. -3 and the ratio is 2.

17. 64 and the ratio is $\frac{1}{2}$.

18. -243 and the ratio is $\frac{1}{3}$.

19. 2 and the ratio is $\sqrt{3}$.

117. The *n*th term of a geometrical progression. Since by definition any geometrical progression is represented by

$$a,\ ar,\ ar^2,\ ar^3,\ \cdots,$$

it is evident that the exponent of r in any term is 1 less than the number of the term. Therefore, if t_n denotes the nth, or general, term of any geometrical progression,

$$t_n = ar^{n-1}.$$

EXERCISES

1. Find the fifth term of $4, 12, 36, \cdots$.

Solution. Here $a = 4$, $r = 3$, $n - 1 = 4$.
Substituting these values in the formula $t_n = ar^{n-1}$,

$$t_4 = 4 \cdot 3^4 = 324.$$

2. Find the eighth term of $1, 2, 4, \cdots$.

3. Find the tenth term of $3, 6, 12, \cdots$.

4. Find the eighth term of $3, 2, \frac{4}{3}, \cdots$.

5. Find the twelfth term of $7, -14, 28, \cdots$.

6. Find t_9 of $15, -5, +\frac{5}{3}, \cdots$.

7. Find t_{12} of $12a^2, 9a^2, \dfrac{27a^2}{4}, \cdots$.

8. Find t_6 of $\dfrac{-2c^3}{5}, -1, \dfrac{-5}{2c^3}, \cdots$.

9. Find t_8 of $4\sqrt{2}, 4, 2\sqrt{2}, \cdots$.

10. Find t_7 of $\dfrac{1}{2\sqrt{2}}, \dfrac{1}{2}, \dfrac{\sqrt{2}}{2}, \cdots$.

11. Write the twentieth term of \$4.12(1.01), \$4.12$(1.01)^2$, \$4.12$(1.01)^3$, $\cdots$.

12. Since the nth term of a geometrical progression is ar^{n-1}, what is the $(n-1)$th term? the $(n-2)$th? the $(n-3)$th? the $(n+1)$th? the $(n+2)$th?

13. The first and second terms of a geometrical progression are m and n respectively. Find the next two terms.

118. Geometrical means. Geometrical means between two numbers are numbers which form, with the two given ones as the first and the last term, a geometrical progression.

Thus ar and ar^2 are the geometrical means in a, ar, ar^2, ar^3.

EXAMPLES

1. Insert two real geometrical means between 9 and 72.

Solution. There are four terms in the geometrical progression, and $a = 9$, $n = 4$, and $t_n = t_4 = 72$.

Substituting these values in $t_n = ar^{n-1}$,

$$72 = 9\,r^3.$$

Whence $$r = 2, \text{ and } -1 \pm \sqrt{-3}.$$

The required geometrical progression is 9, 18, 36, 72.

2. Insert one geometrical mean between h and k.

Solution. There are three terms in the progression, and $a = h$, $n = 3$, and $t_n = k$.

Substituting these values in $t_n = ar^{n-1}$, we have

$$k = h \cdot r^2.$$

Solving, $$r = \pm\sqrt{\frac{k}{h}}.$$

Hence the progression is $k, \pm h\sqrt{\frac{k}{h}}, k$, or $h, \pm\sqrt{hk}, k$.

It follows from the above that the geometrical mean between two numbers is the square root of their product.

ORAL EXERCISES

Insert one geometrical mean between:

1. 1 and a^2.
2. 1 and x^4.
3. 1 and $4x^2$.
4. 3 and 75.
5. a and a^3.
6. a^5 and a^9.

Insert two geometrical means between:

7. 1 and x^3.
8. 1 and 2^3.
9. 1 and 27.
10. 1 and 125.
11. a and a^4.
12. x and x^7.
13. x^2 and x^8.
14. 2 and 16.
15. 5 and 40.

EXERCISES

Obtain progressions involving real terms only:

1. Insert two geometrical means between 21 and 168.

2. Insert two geometrical means between 15 and 405.

3. Insert three geometrical means between 3 and 243.

4. Insert one geometrical mean between 9 and 81.

5. Insert one geometrical mean between a^{12} and a^{20}.

6. Insert three geometrical means between -9 and -144.

7. The fourth term of a geometrical progression is 16, the eighth term is 256. Find the tenth term.

8. The second term of a geometrical progression is $4\sqrt{3}$, the fifth term is $\frac{4}{9}$. Find the first term and the ratio.

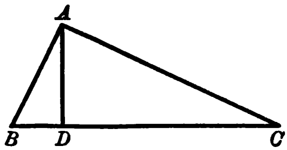
9. Show that the geometrical means between a and c are $\pm\sqrt{ac}$.

10. The first and fourth terms of a geometrical progression are a and c respectively. Find the second and third terms.

11. Insert three geometrical means between h and k.

12. The sum of the first and fourth terms of a geometrical progression is 56. The second term is 6. Find the four terms.

13. In the accompanying figure, ABC is a right triangle and AD is perpendicular to the hypotenuse BC. Under these conditions the length of AD is always a geometrical mean between the lengths of BD and DC.

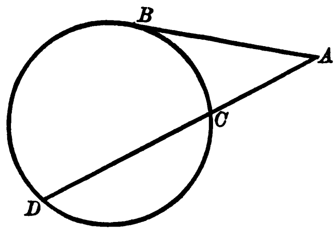


(*a*) If $BD = 4$ and $DC = 9$, find AD.

(*b*) If $BD = 3$ and $BC = 21$, find AD.

(*c*) If $BC = 25$ and $AD = 10$, find BD and DC.

14. In the accompanying figure, AB touches and AD cuts the circle. Under such conditions the length of AB is always a geometrical mean between the lengths of AC and AD.



(*a*) If $AD = 36$ and $AC = 4$, find AB.

(*b*) If $DC = 90$ and $AB = 24$, find AC and AD.

119. Geometrical series. The indicated sum of n terms of a geometrical progression is called a **geometrical series.** The process of obtaining in its simplest form the expression for this sum is often called finding the sum of the series.

The expression for the sum is derived as follows:

Let S_n denote the indicated sum.

$$S_n = a + ar + ar^2 + \cdots + ar^{n-3} + ar^{n-2} + ar^{n-1}$$
$$= a(1 + r + r^2 + \cdots + r^{n-3} + r^{n-2} + r^{n-1}). \qquad (1)$$

$$S_n = a\left(\frac{1-r^n}{1-r}\right), \qquad (2)$$

since the polynomial in (1) is equal to the fraction in the parenthesis of (2) (see Exercise 18, p. 39).

Hence $$S_n = \frac{a - ar^n}{1-r}.$$

EXERCISES

1. Find the sum of the first nine terms of 3, -6, 12, $\cdots$.

Solution. $$S_n = \frac{a - ar^n}{1 - r}.$$

By the conditions, $a = 3$, $r = -2$, and $n = 9$.

Substituting, $$S_9 = \frac{3 - 3(-2)^9}{1-(-2)} = 513.$$

2. Find the sum of 1, 5, 25, $\cdots$ to nine terms.
3. Find S_8 for 2, -4, 8, $\cdots$.
4. Find S_7 for 40, 20, 10, $\cdots$.
5. Find S_8 for -180, 90, -45, $\cdots$.
6. Find S_6 for $\frac{2}{3}$, 1, $\frac{3}{2}$, $\cdots$.
7. Find S_{10} for a^8, a^5, a^7, $\cdots$.
8. Find S_7 for $2\sqrt{3}$, 6, $6\sqrt{3}$, $\cdots$.
9. Find S_n for 4, 12, 36, $\cdots$.
10. Find S_n for 125, $-25\sqrt{5}$, 25, $\cdots$.
11. Find S_{n-1} for 3, 12, 48, $\cdots$.
12. Find S_n for 3, -15, 75, $\cdots$.
13. Find S_{n-2} for x, $4x^4$, $16x^7$, $\cdots$.
14. Show that for a geometrical progression $S_n = \dfrac{a - rl}{1 - r}$.

HINT. Substitute l for the factor ar^{n-1} in the last term of the numerator in the formula of Exercise 1.

15. A rubber ball falls from a height of 60 inches and on each rebound rises 50% of the previous height. How far does it fall on its sixth descent? Through what distance has it moved at the end of the sixth descent?

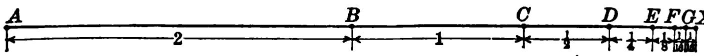
120. Infinite geometrical series. If the number of terms of a geometrical series is unlimited, it is called an **infinite geometrical series.**

In the progression 2, 4, 8, $\cdots$ the ratio is positive and greater than 1, and each term is greater than the term preceding it. Such a progression is said to be increasing. Obviously the sum of an unlimited number of terms in such an increasing geometrical progression is unlimited. If $r > 1$ (read "r is greater than 1"), the sum can be made as large as we please by taking enough terms.

In the progression 2, 1, $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$, $\cdots$ the ratio is positive and less than 1, and each term is less than the term preceding it. Such a progression is said to be decreasing. Though the number of terms of such a geometrical progression be unlimited, the sum of as many terms as we choose to take is always less than some definite number.

The sum of the first three terms of the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \cdots$ is $3\frac{1}{2}$; of four terms, is $3\frac{3}{4}$; of five terms, is $3\frac{7}{8}$; of six terms, is $3\frac{15}{16}$; of seven terms, is $3\frac{31}{32}$, etc. Here for any number of terms the sum is always less than 4.

The limit of the sum of the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ can be seen by reference to the following diagram:



Here the terms of the above series are represented by intervals on the line AX, which is four units in length; that is, $AB = 2$, $BC = 1$, $CD = \frac{1}{2}$, $DE = \frac{1}{4}$, $EF = \frac{1}{8}$, $FG = \frac{1}{16}$, etc.

A study of the several intervals reveals the fact that as we pass from A toward X each new interval is just one half of the then unused portion of AX. It follows, then, that the distance laid off gradually approaches, though it never equals, AX, for at no time is an interval adjoined which is more than one half of that which then remained. Therefore there will always be an unused interval between the extremity of the last interval used and the point X.

Since the sum of the intervals laid off approaches but never quite equals the interval AX, we have in the above diagram an illustration of the fact that the sum of the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots$ approaches, yet always remains less than, 4.

The formula for the sum of a geometric series $S_n = \frac{a - ar^n}{1-r}$ may be written as the difference of two fractions,

$$S_n = \frac{a}{1-r} - \frac{ar^n}{1-r}.$$

For the series $2 + 1 + \frac{1}{2} + \frac{1}{4} + \cdots$ it follows that

$$S_n = \frac{2}{1-\frac{1}{2}} - \frac{2(\frac{1}{2})^n}{1-\frac{1}{2}}. \qquad (3)$$

Now $(\frac{1}{2})^2 = \frac{1}{4}$, $(\frac{1}{2})^3 = \frac{1}{8}$, $(\frac{1}{2})^4 = \frac{1}{16}$, $(\frac{1}{2})^5 = \frac{1}{32}$. Consequently $(\frac{1}{2})^n$ becomes very small if n is taken very great. Therefore $2(\frac{1}{2})^n$, the numerator of the last fraction in (3), decreases and approaches zero as n increases *without limit.* Since the denominator of the fraction remains $\frac{1}{2}$ while the numerator approaches zero, the value of the fraction decreases and approaches zero as n increases. Then if S_∞ denotes S_n where n has increased without limit, we may write

$$S_\infty \text{ approaches } \frac{2}{1-\frac{1}{2}}, \text{ or } 4.$$

In general, if $r < 1$ ("if r is numerically less than 1"), the numerical value of fraction $\frac{ar^n}{1-r}$ approaches zero as n increases without limit. Under such conditions the formula

$$S_n = \frac{a}{1-r} - \frac{ar^n}{1-r} \text{ becomes } S_\infty = \frac{a}{1-r}.$$

This means that for r numerically less than 1, S_n approaches $\frac{a}{1-r}$, but for any definite value of n it is always numerically less than this number.

Hence whenever we speak of the sum of such a series we mean the *limit* which the sum approaches as n increases indefinitely.

EXERCISES

Find the sum of the following series:

1. $4 + 1 + \frac{1}{4} + \cdots$.

Solution. $S_\infty = \dfrac{a}{1-r}$.

Substituting, $S_\infty = \dfrac{4}{1-\frac{1}{4}} = 5\frac{1}{3}$.

2. $1 + \frac{1}{2} + \frac{1}{4} + \cdots$.
3. $3 + (-1) + \frac{1}{3} + \cdots$.
4. $5 + (-2) + \frac{4}{5} + \cdots$.
5. $2 + \sqrt{2} + 1 + \cdots$.
6. $5a + \dfrac{5a}{2} + \dfrac{5a}{4} + \cdots$.
7. $1 + x + x^2 + \cdots$. $(x<1)$
8. $5 + \sqrt{5} + 1 + \cdots$.
9. $1 + \dfrac{1}{x} + \dfrac{1}{x^2} + \cdots$. $(x>1)$
10. .121212.

Hint. $.121212 = \frac{12}{100} + \frac{12}{10000} + \frac{12}{1000000} + \cdots$.

11. $.666\cdots$.
12. $.151515\cdots$.
13. $.3232\cdots$.
14. $25.2727\cdots$.
15. $.71515\cdots$.
16. $.3108108\cdots$.

17. A flywheel whose circumference is 5 feet makes 80 revolutions per second. If it makes 99% as many revolutions each second thereafter as it did the preceding second, how far will a point on its rim have moved by the time it is about to stop?

Note. In the study of geometrical progressions we have seen that the sum of the infinite series $1 + x + x^2 + x^3 + \cdots$ is a definite number when x has any value less than 1. But it has no finite value when x is equal to or greater than 1; that is, we have an expression which we cannot use arithmetically unless x has a properly chosen value. If we were studying some problem which involved such a series, it would be a matter of the most vital importance to know whether the values of x under discussion were such as to make the series meaningless.

This question of distinguishing between expressions which converge (that is, the sum of whose terms approaches a limit) and those which do not has an interesting history. Newton and his followers in the seventeenth century dealt with infinite series and always assumed

that they converged, as, in fact, most of them did. But as more complicated series came into use it became more difficult to tell from inspection whether they meant anything or not for a given value of the variable.

It was not until the beginning of the nineteenth century that Gauss, Abel, and Cauchy, in Germany, Norway, and France respectively, began to study this subject effectively and to devise far-reaching tests to determine the values of x for which certain series converge to a finite limit. It is said that on hearing a discussion by Cauchy in regard to series which do not always converge, the astronomer La Place became greatly alarmed lest he had made use of some such series in his great work, "Celestial Mechanics." He hurried home and denied himself to all distractions until he had examined every series in his book. To his intense satisfaction they all converged. In fact it has often been observed that a genius can safely take chances in the use of delicate processes, which seem very foolish and unsafe to a man of ordinary ability.

MISCELLANEOUS EXERCISES

1. The digits of a certain three-digit number are in geometrical progression. The sum of the digits is 13. The number divided by the sum of its digits gives a quotient of 10 and a remainder of 9. Find the number.

2. What per cent of the eleventh term of the progression 1, 2, 4, 8, $\cdots$ is the eleventh term of 1, 101, 201, $\cdots$?

3. Compare the sum of the first ten terms of the first progression in Exercise 2 with the sum of the first ten terms of the second progression.

4. What meaning may be attached to (*a*) $\$500 \cdot (1.06)^8$? (*b*) $\$500 \cdot (1.05)^5$? (*c*) $\$500 \cdot (1.03)^6$?

5. What will \$100 amount to in three years, with interest at 4%, compounded annually? in five years?

6. What will \$100 amount to in two years, with interest at 6%, compounded semiannually? in three years?

7. For each of ten successive years a man saves $200 from his salary and places it in a savings bank where it draws 4% interest, compounded annually. Find the total amount of his savings at the time of his fifth annual deposit. Indicate (without multiplication) the amount of his deposit at the beginning of the eleventh year.

8. A loan of S dollars is to be repaid in four equal annual payments of p dollars each. Find p if money is worth $r\%$.

Solution. The sum due at beginning of second year is

$$S(1+r)-p. \tag{1}$$

The sum due at beginning of third year is

$$[S(1+r)-p](1+r)-p. \tag{2}$$

The sum due at beginning of fourth year is

$$\{[S(1+r)-p](1+r)-p\}(1+r)-p. \tag{3}$$

The sum due at beginning of fifth year is

$$[\{[S(1+r)-p](1+r)-p\}(1+r)-p](1+r)-p. \tag{4}$$

By the conditions of the problem, (4) = 0, for all the debt has then been paid. Setting (4) equal to zero and simplifying,

$$S(1+r)^4-p(1+r)^3-p(1+r)^2-p(1+r)-p=0. \tag{5}$$

Solving (5) for p,

$$p=\frac{S(1+r)^4}{(1+r)^3+(1+r)^2+(1+r)+1}. \tag{6}$$

But the denominator in (6) is a geometrical series whose sum by the formula $S_n=\frac{a-ar^n}{1-r}$ is $\frac{(1+r)^4-1}{r}$.

Substituting this last value for the denominator of (6),

$$p=\frac{Sr(1+r)^4}{(1+r)^4-1}. \tag{7}$$

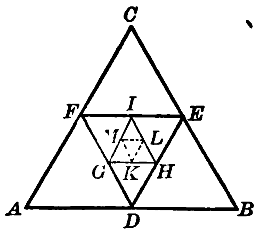
In the general case, if we have n annual payments, the exponent 4 in (7) would be replaced by n, and then $p=\frac{Sr(1+r)^n}{(1+r)^n-1}$.

9. A loan of \$1000 is to be repaid in three equal annual payments, interest at 6%. Find the payment required.

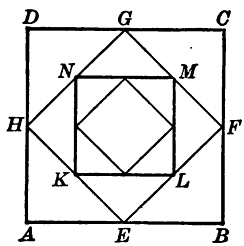
10. A loan of \$2000 bearing interest at 5% is to be repaid in five equal annual payments. Find the payment required.

11. The machinery of a certain mill cost \$10,000. The owner figures that the machinery depreciates 10% in value each successive year. What was the estimate on the value of the machinery at the end of the sixth year?

12. A vessel containing wine was emptied of one third of its contents and then filled with water. This was done four times. What portion of the original contents was then in the vessel?



13. In the adjacent figure the triangle DEF is formed by joining the mid-points of AB, BC, and CA respectively. Triangles GHI and KLM are formed in like manner. If ABC is equilateral, prove that the successive perimeters of the triangles form a geometrical progression. If $AB = 5$, find the sum of the perimeters of all the triangles which may be so formed.



14. Each square in the adjacent figure except the first is formed by joining the mid-points of the square next larger. If $AB = 4$, show that the perimeters of the squares form a decreasing geometrical progression. Find the sum of the perimeters of all the squares which may be so drawn.

CHAPTER XVII

THE BINOMIAL THEOREM

121. Powers of binomials. The following identities are easily obtained by actual multiplication:

$$(a+b)^2 = a^2 + 2\,ab + b^2. \tag{1}$$

$$(a+b)^3 = a^3 + 3\,a^2b + 3\,ab^2 + b^3. \tag{2}$$

$$(a+b)^4 = a^4 + 4\,a^3b + 6\,a^2b^2 + 4\,ab^3 + b^4. \tag{3}$$

$$(a+b)^5 = a^5 + 5\,a^4b + 10\,a^3b^2 + 10\,a^2b^3 + 5\,ab^4 + b^5. \tag{4}$$

If $a+b$ is replaced by $a-b$, the even-numbered terms in each of the preceding expressions will then be negative and the odd-numbered terms will be positive.

122. The expansion of $(a+b)^n$. The form of the expansion for the general case will now be indicated:

The first term is a^n and the last is b^n.

The second term is $+na^{n-1}b$.

The exponents of a decrease by 1 in each term after the first.

The exponents of b increase by 1 in each term after the second.

The product of the coefficient in any term and the exponent of a in that term, divided by the exponent of b increased by 1, gives the coefficient of the next term.

The sign of each term is $+$ if a and b are positive; the sign of each even-numbered term is $-$ if b alone is negative.

According to the above statement we have

$$(a+b)^n = a^n + \frac{n}{1}a^{n-1}b + \frac{n(n-1)}{1\cdot 2}a^{n-2}b^2 + \frac{n(n-1)(n-2)}{1\cdot 2\cdot 3}a^{n-3}b^3 + \ldots + b^n.$$

This expresses in symbols the law known as the **binomial theorem.** The theorem holds for all positive values of n and also, with certain limitations, for negative values.

NOTE. The coefficients of the various terms in the binomial expansion are displayed in a most elegant form as follows:

$$\begin{array}{c} 1 \\ 1\quad 1 \\ 1\quad 2\quad 1 \\ 1\quad 3\quad 3\quad 1 \\ 1\quad 4\quad 6\quad 4\quad 1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \end{array}$$

In this arrangement each row may be derived from the one above it by observing that each number is equal to the sum of the two numbers, one to the right and the other to the left of it, in the line above. Thus $4 = 1 + 3$, $6 = 3 + 3$, etc. The next line is 1 5 10 10 5 1. The successive lines of this table give the coefficients for the expansions of $(a+b)^n$ for the various values of n. The numbers in the last line of the triangle are seen to be the coefficients when $n = 4$; the next line would give those for $n = 5$. This array is known as Pascal's triangle, and was published in 1665. It was probably known to Tartaglia nearly a hundred years before its discovery by Pascal.

ORAL EXERCISES

What is the second term in the expansions of Exercises 1–4?

1. $(a+b)^{18}$. **2.** $(a-b)^{27}$. **3.** $(a+b)^{21}$. **4.** $(a-b)^{31}$.

Assuming that the terms in Exercises 5–10 occur in an expansion of the binomial $a+b$, find (a) the exponents, (b) the coefficient in the next following term of the expansion.

5. $10\,a^2b^3$. **7.** $15\,a^4b^2$. **9.** $252\,a^5b^5$.

6. $3\,ab^2$. **8.** $11\cdot 5\cdot 9\,a^8b^4$. **10.** $20\,a^3b^3$.

EXERCISES

Expand by the rule:

1. $(a+b)^7$.
2. $(a-1)^6$.
3. $(a+1)^7$.
4. $(a+2)^5$.
5. $(a+3)^6$.
6. $(2-a)^7$.

Obtain the first four terms of the following:

7. $(a+b)^{21}$.
8. $(a+b)^{30}$.
9. $(a+1)^{32}$.
10. $(a-2)^{25}$.

Expand:

11. $(a^2+2b)^5$.

HINTS. To avoid confusion of exponents first write

$$(a^2)^5 + \quad (a^2)^4(2b)^1 + \quad (a^2)^3(2b)^2 + \quad (a^2)^2(2b)^3 + \quad (a^2)^1(2b)^4 + (2b)^5.$$

Then in the spaces left for them put in the coefficients according to the rule of this section.

Finally, expand and simplify each term.

12. $(a^2+2)^6$.
13. $(a^2-2b)^7$.
14. $\left(\dfrac{a^2+1}{b}\right)^5$.
15. $\left(\dfrac{a^2-1}{a^3}\right)^6$.

Obtain in simplest form the first four terms of the following:

16. $(a^2+3b)^{20}$.
17. $(a^2-3b^2)^{10}$.
18. $\left(\dfrac{a^2-1}{a}\right)^{30}$.
19. $\left(\dfrac{a}{b}+\dfrac{b}{a}\right)^{20}$.
20. $\left(\dfrac{2x}{y^3}-\dfrac{y^4}{x^5}\right)^8$.
21. $\left(\dfrac{a^2}{b^3}+\dfrac{2b^2}{a^4}\right)^{12}$.
22. $\left(\dfrac{3x^5}{y^3}-\dfrac{2y^{15}}{9x^{12}}\right)^5$.
23. $\left(\dfrac{\sqrt{x}}{y}+\dfrac{\sqrt{y}}{x}\right)^{12}$.
24. $\left(1+\dfrac{1}{n}\right)^n$.

25. Write the first six terms of the expansion of $(a+b)^n$, and evaluate it for $n=1$, $n=2$, $n=3$, $n=4$. How does the number of terms compare with n? What is the value of each coefficient after the $(n+1)$th? Why does the expansion not extend to more than five terms when $n=4$?

Compute the following to two decimal places. (In each case carry the computation far enough to be certain that the terms neglected do not affect the second decimal place.)

26. $(1.1)^{10}$.

HINT. $(1.1)^{10} = (1 + .1)^{10}$ etc.

27. $(5.2)^{8}$.

28. $(1.06)^{6}$.

29. $(.98)^{11}$.

HINT. $(.98)^{11} = (1 - .02)^{11}$ etc.

30. $(4.9)^{9}$.

31. $(2.9)^{8}$.

123. Extraction of roots by use of the binomial expansion. The expansion in section 122 may be verified for any particular integral value of n without difficulty by direct multiplication, as in section 121. But if n has a negative or fractional value, a laborious proof is required to show that the expansion is still valid when a is numerically greater than b. Since none of the factors of the coefficients, as n, $n-1$, $n-2$, vanish for fractional or negative values of n, it appears that for such exponents the expansion is an infinite series.

For example,

$$(a+b)^{\frac{1}{2}} = a^{\frac{1}{2}} + \tfrac{1}{2}a^{-\frac{1}{2}}b - \tfrac{1}{8}\cdot a^{-\frac{3}{2}}b^2 + \tfrac{1}{16}\cdot a^{-\frac{5}{2}}b^3 + \cdots.$$

By giving n the values $\frac{1}{2}$ or $\frac{1}{3}$, one can compute the square root or the cube root of a number to any required degree of accuracy.

In such computations it is desirable to let the number which corresponds to a in the binomial exceed the one corresponding to b by as much as possible and at the same time to have $a^{\frac{1}{n}}$ an integer.

NOTE. The process of extracting the square root and even the cube root by means of the binomial expansion was familiar to the Hindus more than a thousand years ago. The German Stifel (1486–1567) stated the binomial theorem for all powers up to the seventeenth, and also extracted roots of numbers by this method.

EXERCISES

Find the first four terms of the following:

1. $(1+x)^{\frac{1}{2}}$. **3.** $(3-x)^{\frac{1}{2}}$. **5.** $(2+x)^{\frac{1}{3}}$.

2. $(2+x)^{\frac{1}{2}}$. **4.** $(1+x)^{\frac{1}{3}}$. **6.** $(3-x)^{\frac{1}{3}}$.

Find to at least three decimals by the binomial theorem:

7. $(27)^{\frac{1}{2}}$.

Solution.
$$\begin{aligned}(27)^{\frac{1}{2}} &= (25+2)^{\frac{1}{2}}\\ &= 25^{\frac{1}{2}} + \tfrac{1}{2}\cdot 25^{-\frac{1}{2}}\cdot 2 - \tfrac{1}{8}\cdot 25^{-\frac{3}{2}}\cdot 2^2\\ &\qquad + \tfrac{1}{16}\cdot 25^{-\frac{5}{2}}\cdot 2^3 - \cdots\\ &= 5 + .2 - .004 + .00016 \cdots = 5.196+.\end{aligned}$$

It is proved in more advanced books that when the terms of an infinite series are alternately plus and minus, and each term is numerically less than the preceding one, the value of the entire sum from a given term on cannot exceed that term. This fact renders these so-called "alternating series" especially convenient for computation, since a definite limit of error is known at each stage of the computation. In this example the error cannot exceed .00016.

8. $(17)^{\frac{1}{2}}$. **9.** $(28)^{\frac{1}{2}}$. **10.** $(38)^{\frac{1}{2}}$. **11.** $(78)^{\frac{1}{2}}$. **12.** $(125)^{\frac{1}{2}}$.

13. $(61)^{\frac{1}{3}}$.

Solution.
$$\begin{aligned}(61)^{\frac{1}{3}} &= (64-3)^{\frac{1}{3}}\\ &= 64^{\frac{1}{3}} - \tfrac{1}{3}\cdot 64^{-\frac{2}{3}}\cdot 3 - \tfrac{1}{9}\cdot 64^{-\frac{5}{3}}\cdot 3^2\\ &\qquad - \tfrac{5}{81}\cdot 64^{-\frac{8}{3}}\cdot 3^3 + \cdots\\ &= 4 - \tfrac{1}{16} - \tfrac{1}{1024} - \tfrac{5}{196608} + \cdots\\ &= 4 - .0625 - .00097 - .00000259 \cdots = 3.93653-.\end{aligned}$$

Here three terms give the result to five figures.

14. $(79)^{\frac{1}{2}}$.

HINT. $(79)^{\frac{1}{2}} = (81-2)^{\frac{1}{2}} = 81^{\frac{1}{2}} - \frac{1}{2}\cdot 81^{-\frac{1}{2}}\cdot 2 + \cdots$.

Here $(81-2)^{\frac{1}{2}}$ yields more accurate results with fewer terms than does $(64+15)^{\frac{1}{2}}$.

15. $(28)^{\frac{1}{3}}$. **16.** $(66)^{\frac{1}{3}}$. **17.** $(30)^{\frac{1}{3}}$. **18.** $(700)^{\frac{1}{3}}$.

124. The factorial notation. The notation $5!$ or $\underline{|5}$ signifies $1 \cdot 2 \cdot 3 \cdot 4 \cdot 5$, or 120. Also $4! = 1 \cdot 2 \cdot 3 \cdot 4 = 24$.

In general, $n! = 1 \cdot 2 \cdot 3 \cdot 4 \cdots (n-2)(n-1)n$.

The symbol $n!$ or $\underline{|n}$ is read "factorial n."

In the factorial notation the denominators of the fourth and fifth terms of the expansion of $(a+b)^n$ become $3!$ and $4!$ respectively (see formula, p. 228).

EXERCISES

Evaluate:

1. $6!$.
2. $4! \cdot 3!$.
3. $5! \cdot 2!$.
4. $6! \div 2!$.
5. $4! - 3! \cdot 2!$.
6. $n! \div (n-1)!$.

Evaluate $\dfrac{n(n-1)(n-2)\cdots(n-r+2)}{(r-1)!}$, when:

7. $n=7$, $r=5$.
8. $n=15$, $r=8$.
9. $n=21$, $r=12$.
10. $n=20$, $r=15$.
11. $n=18$, $r=17$.
12. $n=10$, $r=11$.

125. The rth term of $(a+b)^n$. According to the binomial theorem the fifth term of the expansion on page 228 is

$$+\frac{n(n-1)(n-2)(n-3)a^{n-4}b^4}{4!}.$$

If we note carefully this term and the directions on page 227, we can write down, from the considerations that follow, any required term without writing other terms of the expansion.

The *denominator of the coefficient* of the fifth term is $4!$ From the law of formation the denominator of the sixth term would be $5!$, of the seventh term $6!$, etc. Consequently in the rth term the denominator of the coefficient would be $(r-1)!$.

The *numerator of the coefficient* of the fifth term contains the product of the four factors $n(n-1)(n-2)(n-3)$. The numerator of the sixth term would contain these four and the additional factor $n-4$. Similarly, the last factor in the numerator of the seventh term would be $n-5$, etc. Hence the last factor in the rth term would be $n-(r-2)$, and the numerator of the coefficient of the rth term is $n(n-1)(n-2)(n-3)\cdots(n-r+2)$.

The *exponent of* a in the fifth term is $n-4$, and in the sixth term it would be $n-5$, etc. Therefore in the rth term the exponent of a is $n-(r-1)$, or $n-r+1$.

The *exponent of* b in the fifth term is 4, in the sixth term is 5, etc. Therefore in the rth term the exponent of b is $r-1$.

The *sign* of any term of the expansion (if n is a positive integer) is plus if the binomial is $a+b$. If the binomial is $a-b$, the terms containing the odd powers of b will be negative. In other words, the sign in such cases depends upon whether the exponent $r-1$ is odd or even.

Hence the rth term (r not equal to 1) of $(a+b)_n$ equals

$$\frac{n(n-1)(n-2)(n-3)\cdots(n-r+2)}{(r-1)!}a^{n-r+1}b^{r-1}. \qquad (1)$$

If we wanted the twelfth term, we would in using (1) substitute 12 for r.

EXERCISES

Write the indicated terms:

1. Fifth term of $(a+b)^{10}$.

Solution. Substituting 10 for n and 5 for r in the formula (1) gives

$$\frac{10\cdot 9\cdot 8\cdot 7}{4!}a^6b^4=\frac{10\cdot 9\cdot 8\cdot 7}{4\cdot 3\cdot 2}a^6b^4$$
$$=210\,a^6b^4.$$

2. Sixth term of $(a+b)^9$.

3. Fourth term of $(a+b)^{20}$.

4. Seventh term of $(a-b)^{10}$.

5. Eighth term of $(a-b)^{15}$.

6. Fourth term of $\left(a+\frac{1}{a}\right)^{30}$.

7. Fifth term of $(a^2-b)^{20}$.

8. Sixth term of $\left(\frac{a}{b}-\frac{b^2}{a}\right)^{18}$.

9. Middle term of $(x^2-x)^{16}$.

10. Seventh term of $\left(\frac{a^2}{b}-\frac{2b^2}{a}\right)^{14}$.

11. Fifth term of $\left(\sqrt{x}-\sqrt{\frac{y}{x}}\right)^{15}$.

Find the coefficient of:

12. x^5 in $(1+x)^{10}$.

13. x^8 in $(1+x^2)^{16}$.

14. x^{15} in $(x^3+1)^{15}$.

15. x^{10} in $(x^2-x^{-1})^{14}$.

NOTE. The binomial theorem occupies a remarkable place in the history of mathematics. By means of it Napier was led to the discovery of logarithms, and its use was of the greatest assistance to Newton in making his most wonderful mathematical discoveries. But to-day the results of Newton and of Napier are explained without even so much as a mention of the binomial theorem, for simpler methods of obtaining these results have been discovered.

It was Newton who first recognized the truth of the theorem, not only for the case where n is a positive integer, which had long been familiar, but for fractional and negative values as well. He did not give a demonstration of the general validity of the binomial development, and none even passably satisfactory was given until that of Euler (1707–1783). The first entirely satisfactory proof of this difficult theorem was given by the brilliant young Norwegian Abel (1802–1829).

CHAPTER XVIII

RATIO, PROPORTION, AND VARIATION

126. Ratio. The **ratio** of one number a to a second number b is the quotient obtained by dividing the first by the second, or $\frac{a}{b}$. The ratio of a to b is also written $a : b$.

It follows from the above that all ratios of two numbers are fractions and all fractions may be regarded as ratios.

Thus $\frac{2}{5}$, $\frac{a}{3d}$, $\frac{a+2}{c-d}$, and $\frac{\sqrt{2}}{\sqrt{3}}$ are ratios as well as fractions.

Since ratios like the above are fractions, operations which may be performed on fractions may be performed on these ratios. Hence the value of a ratio is not changed by multiplying or dividing both numerator (antecedent) and denominator (consequent) by the same number.

Thus $$\frac{a}{b} = \frac{a \cdot x}{b \cdot x} \quad \text{and} \quad \frac{a}{b} = \frac{\frac{a}{y}}{\frac{b}{y}}.$$

EXERCISES

Simplify the following ratios by considering them as fractions and reducing the fractions to lowest terms:

1. $\left(\frac{4a^2 + a - 3}{a^2 - 1}\right) : \left(4 + \frac{4a - 1}{a^2 - a - 2}\right)$.

2. 1 kilometer : 1 mile. (1 kilometer = .62 miles.)

3. 1 liter : 1 quart. (1 liter = .001 cubic meter; 1 quart = $\frac{231}{4}$ cubic inches; and 1 meter = 39.4 inches.)

4. A city lot 100×160 feet : 1 acre. (1 acre = 43,560 square feet.)

5. Area of printed portion of this page : total area of the page.

6. If $24,000 is divided between two men so that the portions received are to each other as $5:7$, how much does each receive?

HINT. Let $5x$ and $7x$ be the required parts.

7. Separate 690 into four parts which are to each other as $2:5:7:9$.

8. Show that $\frac{x}{x+2} < \frac{x+3}{x+5}$ if x is positive.

HINT. Reduce the given fractions to respectively equivalent fractions having a common denominator, then compare the numerators of the fractions so obtained.

9. Arrange the ratios $3:4$ and $7:9$ in decreasing order of magnitude.

127. Proportion. A **proportion** is a statement of equality between two ratios. Four numbers, a, b, c, and d, are in proportion if the ratio of the first pair equals the ratio of the second pair.

This proportion is written

$$a:b=c:d \quad \text{or} \quad \frac{a}{b}=\frac{c}{d}.$$

Though both forms are equations, the second is the more familiar one and for this reason is preferable.

NOTE. By the earlier mathematicians ratios were not treated as if they were numbers, and the equality of two ratios which we know as a proportion was not denoted by the same symbol as other kinds of equality. The usual sign of equality for ratios was $::$, a

notation which was introduced by the Englishman Oughtred in 1631 and was brought into common use by John Wallis about 1686. The sign = was used in this connection by Leibnitz (1646–1716) in Germany, and by the Continental writers generally, while the English clung to Oughtred's notation.

In the proportion $\frac{a}{b} = \frac{c}{d}$ the first and fourth terms (a, d) are called the **extremes**, and the second and third terms (b, c) are called the **means**.

128. Mean proportional. A **mean proportional** between two numbers a and b is the number m if $\frac{a}{m} = \frac{m}{b}$. It follows that $m^2 = ab$, or $m = \pm\sqrt{ab}$.

129. Third proportional. A **third proportional** to two numbers a and b is the number t if $\frac{a}{b} = \frac{b}{t}$.

130. Fourth proportional. A **fourth proportional** to three numbers a, b, and c is the number f if $\frac{a}{b} = \frac{c}{f}$.

131. Test of a proportion. Since a proportion is an equality between two ratios (fractions), it is therefore an equation. Hence *any operation which may be performed on an equation may be performed on a proportion.* (See Axioms, p. 16.)

Thus, in the proportion $\frac{a}{b} = \frac{c}{d}$ both members may be multiplied by bd, giving $ad = bc$. Here the first member is the product of the extremes of the proportion, and the second member is the product of the means.

Therefore *in any proportion the product of the extremes equals the product of the means.*

132. Proportions from equal products. The numbers which occur in a pair of equal products may be used in various ways as the terms of a proportion.

Thus, if $$ad = bc,$$

we may write either $$\frac{a}{b} = \frac{c}{d}, \quad \text{or} \quad \frac{a}{c} = \frac{b}{d}.$$

Proof. If $a \cdot d = b \cdot c$ is divided by bd, we obtain

$$\frac{ad}{bd} = \frac{bc}{bd}, \quad \text{or} \quad \frac{a}{b} = \frac{c}{d}. \tag{1}$$

If $a \cdot d = b \cdot c$ is divided by cd, we obtain

$$\frac{a}{c} = \frac{b}{d}. \tag{2}$$

If the means in (1) are interchanged, (2) is obtained. This process of obtaining (2) from (1) is called **alternation.**

If $a \cdot d = b \cdot c$ is divided by ac, we obtain

$$\frac{b}{a} = \frac{d}{c}. \tag{3}$$

If the fractions in (1) are inverted, (3) is obtained. This process of obtaining (3) from (1) is called **inversion.**

EXERCISES

1. Find a mean proportional between: (*a*) 3 and 27; (*b*) $\frac{4}{7}$ and $\frac{7}{9}$; (*c*) $\frac{81}{28}$ and $\frac{7}{9}$; (*d*) $\frac{x}{y}$ and xy.

2. Find a third proportional to: (*a*) 9 and 6; (*b*) 180 and 60; (*c*) 216 and 36.

3. Find the fourth proportional to: (*a*) 14, 10, and 7; (*b*) 27, 3, and 36; (*c*) 96, 12, and 8.

4. Form three proportions from each of the following equations: (*a*) $5x = 9y$; (*b*) $(a + 3) \cdot 2 = (a + 1) \cdot 3$; (*c*) $x^2 - y^2 = r^2 - s^2$.

5. Write by alternation: (*a*) $\frac{2}{3} = \frac{10}{15}$; (*b*) $\frac{2}{9} = \frac{20}{x}$.

6. Write by inversion: (*a*) $\frac{5}{6} = \frac{10}{12}$; (*b*) $\frac{2}{x} = \frac{12}{30}$.

If $\frac{a}{b}=\frac{c}{d}$, prove the following and state the corresponding theorems in words:

7. $\frac{a}{c}=\frac{b}{d}$.

8. $\frac{b}{a}=\frac{d}{c}$.

9. $\frac{a^n}{b^n}=\frac{c^n}{d^n}$.

10. $\frac{\sqrt[n]{a}}{\sqrt[n]{b}}=\frac{\sqrt[n]{c}}{\sqrt[n]{d}}$.

11. $\frac{a+b}{b}=\frac{c+d}{d}$.

HINT. Add 1 to each member of $\frac{a}{b}=\frac{c}{d}$.

12. $\frac{a+b}{a}=\frac{c+d}{c}$.

HINT. Write $\frac{a}{b}=\frac{c}{d}$ by inversion and apply hint of Exercise 11.

13. $\frac{a-b}{b}=\frac{c-d}{d}$.

14. $\frac{a-b}{a}=\frac{c-d}{c}$.

15. $\frac{a+b}{a-b}=\frac{c+d}{c-d}$.

The proportions given in Exercises 11, 13, and 15 are said to be derived from $a:b=c:d$ by **addition, subtraction,** and **addition and subtraction** respectively.

If $\frac{a}{b}=\frac{c}{d}$, show that the following equalities are true:

16. $\frac{5a}{b}=\frac{5c}{d}$.

17. $\frac{2a}{7b}=\frac{2c}{7d}$.

18. $\frac{a^2}{b^2}=\frac{ac}{bd}$.

19. $\frac{5a+b}{5a-b}=\frac{5c+d}{5c-d}$.

20. $\frac{a^2-b^2}{b^2}=\frac{c^2-d^2}{d^2}$.

21. $\frac{a^2-7b^2}{b^2}=\frac{c^2-7d^2}{d^2}$.

22. $\frac{a^2-b^2}{2ab}=\frac{c^2-d^2}{2cd}$.

23. $\frac{7a^2-3b^2}{5ab}=\frac{7c^2-3d^2}{5cd}$.

24. $\frac{a^2+ab+b^2}{c^2+cd+d^2}=\frac{a^2-ab+b^2}{c^2-cd+d^2}$.

25. In the proof which follows give the reason for each step and state the result as a theorem:

If $\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$, then $\frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}$.

Proof. Setting each of the given ratios above equal to r,

$$\frac{a}{b}=r,\quad \frac{c}{d}=r,\quad \text{and}\quad \frac{e}{f}=r. \qquad (1)$$

Then from (1), $a=br,\quad c=dr,\quad e=fr.$ (2)

Adding in (2), $a+c+e=br+dr+fr.$ (3)

Factoring in (3), $a+c+e=(b+d+f)r.$ (4)

Therefore $\frac{a+c+e}{b+d+f}=r.$ (5)

Hence by (1) and (5), $\frac{a+c+e}{b+d+f}=\frac{a}{b}=\frac{c}{d}=\frac{e}{f}.$

26. If $\frac{a}{b}=\frac{c}{d}=\frac{5}{6}$, show that $\frac{a+5}{b+6}=\frac{c}{d}$, or $\frac{5}{6}$.

Solve, using theorems of proportion:

27. $2x:(x+8)=10:3$.

28. $25:x=x:169$.

29. $5:4=(x-3):(x-4)$.

30. $(15+x):(15-x)=13:17$.

31. $(10+x):(20+3x)=(10-x):-3x$.

32. $\frac{\sqrt{x}+2}{\sqrt{x}-2}=\frac{x+1}{x-7}$.

33. Show that the mean proportional between two numbers is the geometric mean between these numbers.

PROBLEMS

1. The surface of a sphere is $4\pi R^2$. If S represents the surface of a sphere, R its radius, and D its diameter, show that: (*a*) $\frac{S_1}{S_2}=\frac{R_1^2}{R_2^2}$; (*b*) $\frac{S_1}{S_2}=\frac{D_1^2}{D_2^2}$.

2. Find the ratio of the surfaces of two spheres whose radii are in the ratio $1:10$.

3. If the diameter of the earth is 7920 miles and that of Mars is 4230 miles, find the ratio of their surfaces.

4. If the diameter of the moon is 2160 miles, find the ratio of its surface to that of the earth.

5. The volume of a sphere is $\frac{4\pi R^3}{3}$. If V represents the volume of a sphere, R its radius, and D its diameter, show that for any two spheres,

$$\frac{V_1}{V_2} = \frac{R_1^3}{R_2^3} = \frac{D_1^3}{D_2^3}.$$

6. The diameter of the sun is approximately one hundred and nine times the diameter of the earth. Find the ratio of their volumes.

7. From Exercises 3 and 5 find the ratio of the volumes of Mars and the earth.

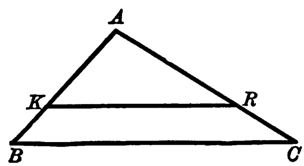
8. Find the ratio of the volumes of the earth and the moon.

9. The areas of two similar triangles are to each other as the squares of any two corresponding lines. If the corresponding sides of two similar triangles are 12 and 20 and the area of the first is 90 square inches, find the area of the second.

10. The areas of two similar triangles are 147 and 300 respectively. If the base of the first is 10, find the corresponding base of the second.

11. If ABC is any triangle and KR is a line parallel to BC, meeting AB at K and AC at R, then

$$\frac{\text{area } ABC}{\text{area } AKR} = \frac{AB^2}{AK^2} = \frac{AC^2}{AR^2} = \frac{BC^2}{KR^2}.$$



If in the accompanying figure area $ABC = 225$ square inches, area $AKR = 81$ square inches, and $AB = 15$ inches, find AK.

12. In the figure of Exercise 11, if $ABC = 845$ square inches, $BC = 13$ inches, and $KR = 7$ inches, find the area AKR.

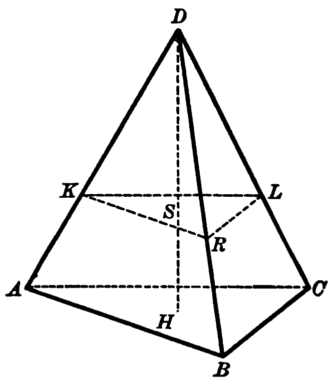
13. If in the figure of Exercise 11 triangle AKR equals $\frac{4}{21}$ of the trapezoid $KBCR$ and $AC = 15$ inches, find AR and RC.

14. In Exercise 13 substitute $\frac{1}{5}$ for $\frac{4}{21}$ and solve for AR to two decimals.

15. If in the figure of Exercise 11 the triangle is equivalent to the trapezoid and $AK = 10$, find KB to two decimals.

16. A certain flagpole casts a shadow 45 feet long at the same time that a near-by post 8 feet high casts a shadow $4\frac{1}{2}$ feet long. Find the height of the pole.

17. The bisector of an angle of a triangle divides the opposite side into segments which are proportional to the adjacent sides. In triangle ABC if $AB = 15$, $BC = 24$, and $CA = 25$, find the segments of BC made by the bisector of angle A.



18. If a plane be passed parallel to the base of a pyramid (or cone), as in the accompanying figure, cutting it in KRL, then pyramid $D - ABC$: pyramid $D - KRL = DH^3 : DS^3$, etc.

If in the adjacent figure the volumes of the pyramids are 8 and 27 cubic inches respectively, and the altitude DH equals 15 inches, find DS.

19. If DH in the accompanying figure is 18 inches and the volume of one pyramid is one third the volume of the other, find DS to two decimals.

20. In the accompanying figure the frustum is seven eighths of the whole pyramid. (*a*) If DH equals 16, find DS; (*b*) if $DH = x$, find DS.

21. If a plane parallel to the base divides the whole pyramid into two parts having equal volumes and $DH = 75$, find to two decimals the parts into which the plane divides DH.

22. The volumes of two similar figures are to each other as the cubes of any two corresponding edges. Compare the volumes of two similar solids, the edge of one of which is 50% greater than the corresponding edge of the other.

23. Compare the radii of two spheres whose volumes are to each other as $125:27$.

133. Variation. The word **quantity** denotes anything which is **measurable**, such as distance, rate, time, and area.

Many operations and problems in mathematics deal with numerical measures of quantities, some of which are fixed and others constantly changing. Such problems as deal with the relation of the numerical measures of at least two changing quantities are called problems in variation.

The theory of variation is really involved in proportion. This fact will become evident after a study of the illustrations of the different kinds of variation here given.

The equation $x = 3y$ may refer to no physical quantities whatever, yet it is possible to imagine y as taking on in succession every possible numerical value, and the value of x as changing with every change of y, and consequently always being three times as great as the corresponding value of y. In this sense, which is strictly mathematical, x and y are variables.

The symbol for variation is $\propto$, and $x \propto y$ is read "x varies directly as y" or "x varies as y."

134. Direct variation. One hundred feet of copper wire of a certain size weighs 32 pounds. Obviously a piece of the same kind 200 feet long would weigh 64 pounds, a piece 300 feet long would weigh 96 pounds, and so on.

Here we have two variables, W (weight) and L (length), so related that the value of W depends on the value of L, and in such a way that W increases proportionately as L increases. That is, W is directly proportional, or merely proportional, to L. Hence, if W_1 and W_2 are any two weights corresponding to the lengths L_1 and L_2 respectively,

$$W_1 : W_2 = L_1 : L_2. \tag{1}$$

The fact expressed by (1) can be stated in the form of a variation, thus: $W \propto L$.

In general, if $x \propto y$, and x and y denote *any* two corresponding values of the variables, and x_1 and y_1 a *particular* pair of corresponding values of these variables,

then $$\frac{x}{x_1} = \frac{y}{y_1}. \tag{2}$$

From (2), $$x = \left(\frac{x_1}{y_1}\right)y. \tag{3}$$

But $\frac{x_1}{y_1}$ is a constant, being the quotient of two definite numbers.

Call this constant K, and (3) may be written

$$x = Ky.$$

That is, *if one variable varies as a second, the first always equals the second multiplied by some constant.*

Thus for the copper wire just mentioned, $W = \frac{32}{100}L$, or $\frac{8}{25}L$. Here, though W varies as L varies, W is always equal to L multiplied by the constant $\frac{8}{25}$.

EXERCISES

1. If $x \propto y$, and $x = 3$ when $y = 8$, find x when $y = 12$.

Solution. The variation is direct.

Therefore $$\frac{3}{x} = \frac{8}{12}.$$

Solving, $$x = 4\tfrac{1}{2}.$$

2. If $x \propto y$, and $x = 8$ when $y = 15$, find y when $x = 10$.

3. If $x \propto y$, and $x = h$ when $y = k$, find y when $x = r$.

4. If $x \propto y$, and $x = 2$ when $y = 5$, find K.

135. Inverse variation. If a tank full of water is emptied in 24 minutes through a smooth outlet in which the area of the opening A is 1 square inch, an outlet in which A is 2 square inches would empty the tank in one half the time, or in 12 minutes; and an outlet in which A is 3 square inches would empty the tank in 8 minutes.

Suppose it possible to increase or decrease A at will. When A is doubled t is halved; when A is trebled t is divided by 3; and so on. We then have in t (the time required to empty the tank) and in A (the area of the opening) two related variables such that if A increases, t will decrease proportionally, while if A decreases, t will increase proportionally.

Now let t_1 and t_2 be *any two* times corresponding to the areas A_1 and A_2 respectively; then

$$t_1 : t_2 = A_2 : A_1. \tag{1}$$

The letters and the subscripts in (1) say: *The first time is to the second time as the second area is to the first area.*

The proportion (1) may be written $t_1 : t_2 = \dfrac{1}{A_1} : \dfrac{1}{A_2}$ where the subscripts on the t's and those on the A's come in the same order.

First, from (1), $$t_1 \cdot A_1 = t_2 \cdot A_2. \tag{2}$$

Dividing (2) by A_1A_2, $$\frac{t_1}{A_2} = \frac{t_2}{A_1}. \tag{3}$$

Whence $$t_1\left(\frac{1}{A_2}\right) = t_2\left(\frac{1}{A_1}\right). \tag{4}$$

Therefore
$$t_1 : t_2 = \frac{1}{A_1} : \frac{1}{A_2}. \qquad (5)$$

In the form of a variation (5) becomes $t \propto \frac{1}{A}$.

In general, x varies **inversely** as y when x varies as the **reciprocal** of y; that is,

$$x \propto \frac{1}{y}. \qquad (6)$$

And if x and y denote any two corresponding values of the variable, and x_1 and y_1 a particular pair of corresponding values,

$$x : x_1 = \frac{1}{y} : \frac{1}{y_1}. \qquad (7)$$

Whence
$$\frac{x}{y_1} = \frac{x_1}{y}, \text{ or } xy = x_1y_1. \qquad (8)$$

But x_1y_1 is a constant, being the product of two definite numbers. Call this constant K.

Then (8) becomes $\qquad xy = K.$

That is, *if one variable varies inversely as another, the product of the two is a constant.*

EXERCISES

1. If x varies inversely as y, and $x = 8$ when $y = 5$, find x when $y = 15$.

Solution. The variation is inverse.

Hence
$$8 : x = \frac{1}{5} : \frac{1}{15}.$$

Solving,
$$x = 2\tfrac{2}{3}.$$

2. If $x \propto \frac{1}{y}$, and $x = 1$ when $y = 25$, find x when $y = 10$.

3. If $y \propto \frac{1}{z}$, and $y = h$ when $z = k$, find y when $z = r$.

4. If $m \propto \frac{1}{n}$, and $m = 2$ when $n = \frac{1}{3}$, find m when $n = 12$.

5. If $t \propto \frac{1}{n}$, and $t = 2$ when $n = 8$, find n when $t = 8$.

6. If $w \propto \frac{1}{d}$, and $w = 100$ when $d = 4000$, find w when $d = 5000$.

7. If $t \propto \frac{1}{r}$, and $t = 4$ when $r = 25$, find K.

8. If $w \propto \frac{1}{d}$, and $w = 200$ when $d = 4000$, find K.

136. Joint variation. If the base of a triangle remains constant while the altitude varies, the area will vary as the altitude. Similarly, if the base varies while the altitude remains constant, the area will vary as the base. If both base and altitude vary, the area varies as the product of the two; that is, the area of the triangle varies **jointly** as the base and altitude. Further, if at any time A denotes the area of a variable triangle, and h_1 and b_1 the corresponding altitude and base, then

$$A_1 = \frac{h_1 b_1}{2}. \qquad (1)$$

If A_2 denotes the area at *any other* time, and h_2 and b_2 the corresponding altitude and base, then

$$A_2 = \frac{h_2 b_2}{2}. \qquad (2)$$

Now $(1) \div (2)$ gives $A_1 : A_2 = h_1 b_1 : h_2 b_2$.
In the form of a variation this last proportion becomes

$$A \propto hb.$$

In general, any variable x varies jointly as two others, y and z, if

$$x \propto yz; \tag{1}$$

that is, if x varies as the product of the two.

If x varies jointly as y and z, and if x, y, and z denote *any* corresponding values of the variables, while x_1, y_1, and z_1 denote a *particular* set of such values, then

$$\frac{x}{x_1} = \frac{yz}{y_1 z_1}. \tag{2}$$

From (2),

$$x = \left(\frac{x_1}{y_1 z_1}\right) yz. \tag{3}$$

But in (3) the fraction $\frac{x_1}{y_1 z_1}$ is a oonstant, since x_1, y_1, and z_1 are particular values of the variables x, y, and z. Calling this constant K, we may write $x \propto yz$ as the equation

$$x = Kyz.$$

One variable may vary directly as one variable (or several variables) and inversely as another (or several others). Also one variable may vary as the square, or the cube, or the square root, or the reciprocal, or as any algebraic expression whatever involving the other variable (or variables).

EXERCISES

1. If x varies jointly as y and z, and $x = 24$ when $y = 6$ and $z = 8$, find x when $y = 18$ and $z = 4$.

Solution. The variation is joint.

Therefore $$\frac{24}{x} = \frac{6 \cdot 8}{18 \cdot 4}.$$

Solving, $$x = 36.$$

2. If x varies as yz, and $x = 10$ when $y = 15$ and $z = 6$, find x when $y = 9$ and $z = 8$.

3. If $A \propto hb$, and $A = 30$ when $h = 5$ and $b = 12$, find A when $h = 7$ and $b = 10$.

4. If $A \propto hb$, and $A = 48$ when $h = 8$ and $b = 12$, find K.

5. If x varies directly as y and inversely as z, and $x = 10$ when $y = 5$ and $z = 27$, find x when $y = 12$ and $z = 36$.

6. If V varies directly as T and inversely as P, and $V = 80$ when $P = 30$ and $T = 300$, find P when $T = 400$ and $V = 40$.

PROBLEMS

1. The weight of any object below the surface of the earth varies directly as its distance from the center. An object weighs 172 pounds at the surface of the earth. What would be its weight (*a*) 1000 miles below the surface? (*b*) 3000 miles below the surface? (*c*) at the center of the earth? (Radius of the earth = 4000 miles.)

2. The distance which sound travels varies directly as the time. A soldier measures with a stop watch the time elapsing between the sight of the flash of an enemy's gun and the sound of its report. If sound travels 1100 feet per second, how far off was the enemy when the observed time was $5\frac{2}{5}$ seconds?

3. When the volume of air in a bicycle pump is 24 cubic inches, the pressure on the handle is 30 pounds. Later when the volume of air is 20 cubic inches, the pressure is 36 pounds. Assume that a proportion exists here, determine whether it is direct or inverse, and find the volume of the air when the pressure is 42 pounds.

4. The distance (in feet) through which a body falls from rest varies as the square of the time in seconds. If a body falls 16 feet in 1 second, how far will it fall in (*a*) 3 seconds? (*b*) 10 seconds?

5. The intensity (brightness) of light varies inversely as the square of the distance from the source of the light. A reader holds his book 3 feet from a lamp and later 6 feet distant. At which distance does the page appear the brighter? How many times as bright?

6. A lamp shines on the page of a book 5 feet distant. Where must the book be held so that the page will receive twice as much light? four times as much light?

7. The area of a circle varies as the square of its radius. The area of a certain circle is 154 square inches and its radius is 7 inches. Find the radius of a circle whose area is 616 square inches.

8. The area illuminated on a screen by a spot light varies directly as the square of the distance from the source of the light to the screen. If the lighted area at a distance of 40 feet is a circle of diameter 10 feet, find the diameter of the illuminated circle at a distance of 15 feet.

9. The weight of an object above the surface of the earth varies inversely as the square of its distance from the center of the earth. An object weighs 172 pounds at the surface of the earth. What would it weigh (*a*) 1000 miles above the surface? (*b*) 3000 miles above the surface? (*c*) 5000 miles above the surface?

10. How far above the surface of the earth would a 150-pound object have to be placed so that its weight would be reduced one third?

11. The weight of a sphere of given material varies directly as the cube of its radius. Two spheres of the same material have radii 3 inches and 5 inches respectively. The first weighs 8 pounds. Find the weight of the second.

12. The time required by a pendulum to make one vibration varies directly as the square root of its length. If a pendulum 100 centimeters long vibrates once in 1 second, find the time of one vibration of a pendulum 81 centimeters long.

13. Find the length of a pendulum which vibrates once in 2 seconds; once in 7 seconds.

14. The pressure of wind on a plane surface varies jointly as the area of the surface and the square of the wind's velocity. The pressure on 1 square foot is .9 pound when the rate of the wind is 15 miles per hour. Find the velocity of the wind when the pressure on 1 square yard is 72.9 pounds.

15. The pressure of water on the bottom of a containing vessel varies jointly with the area of the bottom and the depth of the water. When the water is 1 foot deep the pressure on 1 square foot of the bottom is 62.5 pounds. Find the pressure on the bottom of a circular tank of 14 feet diameter in which the water is 10 feet deep.

16. The cost of ties for a railroad varies directly as the length of the road and inversely as the distance between the ties. The cost of ties for a certain piece of road, the ties being 2 feet apart, was $1320. Find the cost of ties for a piece twenty times as long as the first if the ties are $2\frac{1}{4}$ feet apart.

CHAPTER XIX

LOGARITHMS

137. Introduction. **Logarithms** were invented to shorten the work of extended numerical computations which involve one or more of the operations of multiplication, division, involution, and evolution. Their use has decreased the labor of computing to such an extent that many calculations which would require hours without the use of logarithms can be performed with their aid in one tenth of the time or less.

138. Definition of logarithm and base. If we write the equation

$$n = b^l, \tag{1}$$

we express therein the essential relation between a **number**, n, and its **logarithm**, l, for a given **base**, b. In the notation of logarithms this is written

$$\log_b n = l, \tag{2}$$

and it is read "the logarithm of n to the base b equals l."

We can define verbally in one statement both logarithm and base as follows:

*The **logarithm** of a given number is the exponent in the power to which another number, called the **base**, must be raised in order to equal the given number.*

It is important to realize that equations (1) and (2) are merely two different ways of expressing precisely the same relations, one the exponential way, the other the logarithmic.

Above all it is necessary to keep in mind the fact that *a logarithm is an exponent.*

Thus in $32 = 2^5$ the given number is 32, the base is 2, and the logarithm is 5; that is, $\log_2 32 = 5$.

139. Systems of logarithms. The base of the *common*, or *Briggs*, system of logarithms is 10. Hence a table of common logarithms is really a table of exponents of the number 10. Since the greater portion of these exponents are approximate values of irrational numbers, it follows that computations by means of logarithms give only approximate results. Tables exist, however, in which each logarithm is given to twenty or more decimals; hence practically any desired degree of accuracy can be obtained by using the proper table. This system is used in numerical work almost exclusively. The table on pages 266–267 is a table of common logarithms carried to four decimal places.

The only other system of logarithms used in computations is called the *natural system.* It has for its base the irrational number 2.7182+, which is usually denoted by the letter e and is used mainly for theoretical purposes.

It can be proved that the laws given on pages 91–92, governing the use of rational exponents, hold for irrational exponents. In the work on logarithms this fact will be assumed.

ORAL EXERCISES

1. If $8 = 2^x$, $x = ?$ $\log_2 8 = ?$

2. If $1000 = 10^x$, $x = ?$ $\log_{10} 1000 = ?$

3. $\log_2 16 = ?$ $\log_3 81 = ?$ $\log_5 625 = ?$

4. If $b^3 = 64$, $b = ?$ If $b^5 = 243$, $b = ?$

5. $10^1 = ?$ $\log_{10} 10 = ?$

6. $10^2 = ?$ $\log_{10} 100 = ?$

7. $10^0 = ?$ $\log_{10} 1 = ?$

8. $3^0 = ?$ $\log_3 1 = ?$

9. $5^0 = ?$ $\log_5 1 = ?$

10. $n^0 = ?$ $\log_n 1 = ?$

	Number	Base	Logarithm		Number	Base	Logarithm
11.	8	2	?	**24.**	?	4	3
12.	32	2	?	**25.**	?	6	2
13.	64	4	?	**26.**	9	?	$\frac{1}{2}$
14.	125	5	?	**27.**	8	?	$\frac{3}{2}$
15.	1000	10	?	**28.**	16	?	$\frac{3}{4}$
16.	16	?	2	**29.**	2	4	?
17.	81	?	2	**30.**	3	9	?
18.	81	?	4	**31.**	2	8	?
19.	49	?	2	**32.**	3	81	?
20.	216	?	3	**33.**	7	49	?
21.	?	10	3	**34.**	8	4	?
22.	?	10	2	**35.**	4	8	?
23.	?	10	1	**36.**	.1	10	?

Read in the notation of logarithms:

37. $300 = 10^{2.48}$.

38. $65 = 10^{1.81}$.

39. $4 = 10^{.60}$.

40. $1 = 10^{0}$.

41. $.10 = 10^{-1}$.

42. $1730 = 10^{3.238}$.

43. $173 = 10^{2.238}$.

44. $1.73 = 10^{.238}$.

45. $.173 = 10^{-1+.238}$.

46. $.0173 = 10^{-2+.238}$.

Read Exercises 47–49 and 54–56 as powers of 10:

47. $\log 3 = .48$. **48.** $\log 20 = 1.301$. **49.** $\log 4.9 = .69$.

50. $\log_{10}100 + \log_{10}1000 + \log_{10}10{,}000 = ?$

51. $\log_{10}10 + \log_{10}.01 - \log_{10}1 = ?$

52. $\log_2 8 + \log_3 27 + \log_4 1 = ?$

53. $\log_3 9 + \log_4 64 = ?$

54. $\log 490 = 2.69$.

55. $\log .0049 = -3 + .69$.

56. $\log 381 = 2.58$.

BIOGRAPHICAL NOTE. *John Napier.* Although many scientists have been honored with titles on account of their discoveries, very few of the titled aristocracy have become distinguished for their mathematical achievements. A notable exception to this rule is found in John Napier, Lord of Merchiston (1550–1617), who devoted most of his life to the problem of simplifying arithmetical operations. Napier was a man of wide intellectual interests and great activity. In connection with the management of his estate he applied himself most seriously to the study of agriculture, and experimented with various kinds of fertilizers in a somewhat scientific manner, in order to find the most effective means of reclaiming soil. He spent several years in theological writing. When the danger of an invasion by Philip of Spain was imminent he invented several devices of war. Among these were powerful burning mirrors, and a sort of round musket-proof chariot, the motion of which was controlled by those within, and from which guns could be discharged through little portholes.

But by far the most serious activity of Napier's life was the effort to shorten the more tedious arithmetical processes. He invented the first approximation to a computing machine, and also devised a set of rods, often called Napier's bones, which were of assistance in multiplication. His crowning achievement, however, was the invention of logarithms, to which he devoted fully twenty years of his life.

140. Steps preceding computation. Before computation by means of the table can be taken up, two processes requiring considerable explanation and practice must be mastered.

I. To find from the table the logarithm of a given number.

II. To find from the table the number corresponding to a given logarithm.

141. Characteristic and mantissa. Unless a number is an exact power of 10, its logarithm consists of an *integer* and a *decimal.*

This fact is illustrated in Exercises 37–46, p. 254.

The integral part of a logarithm is called its **characteristic.**

The decimal part of a logarithm is called its **mantissa.**

Log 200 = 2.301. Here 2 is the characteristic and .301 is the mantissa.

The characteristic of any number is obtained not from a table of logarithms but by an inspection of the number itself, according to rules which will now be derived.

$$\begin{array}{lll}
10^4 = 10{,}000; & \text{that is, the log } 10{,}000 & = 4. \\
10^3 = 1000; & \text{that is, the log } 1000 & = 3. \\
10^2 = 100; & \text{that is, the log } 100 & = 2. \\
10^1 = 10; & \text{that is, the log } 10 & = 1. \\
10^0 = 1; & \text{that is, the log } 1 & = 0. \\
10^{-1} = .1; & \text{that is, the log } .1 & = -1. \\
10^{-2} = .01; & \text{that is, the log } .01 & = -2. \\
10^{-3} = .001; & \text{that is, the log } .001 & = -3.
\end{array}$$

The preceding table indicates between what two integers the logarithm of a number less than 10,000 lies. This determines the characteristic.

Since 542 lies between 100 and 1000 (that is, between 10^2 and 10^3), log 542 must lie between 2 and 3 and must equal 2 (characteristic) plus a decimal (mantissa).

And since .0045 lies between .001 and .01 (that is, between 10^{-3} and 10^{-2}), log $.0045 = -3$ plus a positive decimal or -2 plus a negative decimal.

For the determination of the characteristic of a positive number we have the following rules:

I. The characteristic of a number greater than 1 is one less than the number of digits to the left of the decimal point.

II. The characteristic of a number less than 1 is negative and numerically one greater than the number of zeros between the decimal point and the first significant figure.

Accordingly the characteristic of 2536 is 3; of 6 is 0; of .4 is -1; of .032 is -2; of .00036 is -4.

JOHN NAPIER

ORAL EXERCISES

What is the characteristic of the following:

1. 347. 2	**5.** 35. 1	**9.** 97.2. 1	**13.** .00972. -3
2. 9864. 3	**6.** 972. 2	**10.** 9.72. 0	**14.** 30.467. 1
3. 95. 1	**7.** 9720. 3	**11.** .972. -1	**15.** .5000. -1
4. 7. 0	**8.** 97200. 4	**12.** .0972. -2	**16.** .000375. -4

The table on pages 266–267 gives the mantissas of numbers from 10 to 999. Before each mantissa a decimal point is understood.

The numbers 5420, 542, 5.42, .0542, and .000542 are spoken of as composed of the same significant digits in the same order. They differ only in the position of the decimal point, and consequently their logarithms *to the base 10* will have different characteristics, but they will have the same mantissa.

The last two points are easily illustrated by any two numbers which have the same significant digits in the same order.

$$\log 5.42 = .734, \text{ or } 5.42 = 10^{.734};$$

$$5.42 \cdot 10^2 = 542 = 10^{.734} \cdot 10^2 = 10^{2.734}.$$

Therefore $\log 542 = 2.734$.

The property just explained does not belong to a system of logarithms in which the base is any number other than 10. Thus, if the base is 100, the most convenient number after 10, the logarithms of 5420, 542, 54.2, and 5.42 are respectively 1.8670, 1.3670, .8670, and .3670. While a certain regularity in characteristic and mantissa can be seen here, it is obvious that the rules for obtaining them would not be so simple as they are for the base 10. Moreover, it can be seen that tables of a given accuracy are far shorter when the base is 10 than they would be with any other base.

142. Use of the table. To obtain the logarithm of a number of three or fewer significant figures from the table, we have the

Rule. *Determine the characteristic by inspection.*

Find in column N *the first two significant figures of the given number. In the row with these and in the column headed by the third figure of the number, find the required mantissa.*

ORAL EXERCISES

Find the logarithm of the following:

1. 263.	**4.** 56.	**7.** 3.7.	**10.** 7.
2. 375.	**5.** 560.	**8.** 3700.	**11.** 932.
3. 729.	**6.** 37.	**9.** 5.	**12.** .932.

Solution. The characteristic of .932 is -1 and the mantissa is .9694. Hence $\log .932 = -1 + .9694$. This is usually written in the abbreviated form, $\bar{1}.9694$. The mantissa is always kept positive in order to avoid the addition and subtraction of both positive and negative decimals, which in ordinary practice contain from three to five figures. Negative characteristics, being integers, are comparatively easy to take care of. (The student should note that log .932 is really negative, being $-1 + .9694$, or $-.0306$.)

13. .563.	**15.** .00376.	**17.** .0202.	**19.** 3.86.
14. .0637.	**16.** .00468.	**18.** 725000.	**20.** .987.

143. Interpolation. The process of finding the logarithm of a number not found in the table, from the logarithms of two numbers which are found there, or the reverse of this process, is called **interpolation.**

If we desire the logarithm of a number not in the table, say 7635, we know that it lies between the logarithms of 7630 and 7640, which are given in the table. Since 7635 is halfway between 7630 and 7640, we assume, though it is not *strictly* true, that the required logarithm is halfway

between their logarithms, 3.8825 and 3.8831. In order to find log 7635 we first look up log 7630 and log 7640 and then take half (or .5) their difference (this difference may usually be taken from the column headed D) and add it to log 7630. This gives

$$\log 7635 = 3.8825 + .5 \times .0006 = 3.8828.$$

Were we finding log 7638, we should take .8 of the difference between log 7630 and log 7640 and add it to log 7630 as follows:

$$\begin{aligned} \log 7638 &= 3.8825 + .8 \times .0006 \\ &= 3.8825 + .00048 \\ &= 3.8825 + .0005 \\ &= 3.8830. \end{aligned}$$

Observe that in using four-place tables one should not carry results to five figures. If the fifth figure is 5, 6, 7, 8, or 9, omit it and increase the fourth figure by 1; that is, *obtain results to the nearest figure in the fourth place.*

For finding the logarithm of a number we have the

Rule. *Prefix the proper characteristic to the mantissa of the first three significant figures.*

Then multiply the difference between this mantissa and the next greater mantissa in the table (called the tabular difference) by the remaining figures of the number preceded by a decimal point.

Add the product to the logarithm of the first three figures, taking the nearest decimal in the fourth place.

In this method of interpolation we have assumed that the increase in the logarithm is directly proportional to the increase in the number. As has been said, this is not strictly true, yet the results here obtained are nearly always correct to the fourth decimal place.

EXERCISES

Find the logarithm of the following:

1. 3625.	**5.** 646.8.	**9.** 705.50.
2. 464.7.	**6.** 82.543.	**10.** 3.0075.
3. 52.73.	**7.** 10.101.	**11.** .00286.
4. 42.75.	**8.** 500.35.	**12.** .0007777

144. Antilogarithms. An **antilogarithm** responding to a given logarithm. Thus ar

If we desire the antilogarithm of ; say 4.7308, we proceed as follows: Tb found in the *row* which has 53 in co *column* which has 8 at the top. Henc nificant figures of the antilogarithm characteristic is 4, the number must] left of the decimal point. Thus anti]

Therefore, if the mantissa of a gi in the table, its antilogarithm is ob

Rule. *Find the row and the col mantissa lies. In the row found to are in column* N *for the first two antilogarithm and for the third fig of the column in which the mantis*

Place the decimal point as ind

ORAL EXER

Find the antilogarithm of the

1. 3.8768.	**6.** 7.5866 –
2. 1.8035.	HINT. 7.586(
3. .5763.	**7.** 9.2455
4. $\bar{1}$.3747.	**8.** 4.1335
5. $\bar{2}$.7649.	**9.** 5.7875

If the mantissa of a given logarithm, as 1.5271, is not in the table, the antilogarithm is obtained by interpolation as follows:

The mantissa 5271 lies between

.5263, the mantissa of the sequence 336,

and .5276, the mantissa of the sequence 337.

Therefore the antilogarithm of 1.5271 lies between 33.6 and 33.7. Since the tabular difference is 13 and the difference between .5263 and .5271 is 8, the mantissa .5271 lies $\frac{8}{13}$ of the way from .5263 to .5276. Therefore the required antilogarithm lies $\frac{8}{13}$ of the way from 33.6 to 33.7.

Then $\text{antilog}\, 1.5271 = 33.6 + \frac{8}{13} \times .1,$

and $33.6 + .061 = 33.66.$

Therefore when the mantissa is not found in the table we have the

Rule. *Write the number of three figures corresponding to the lesser of two mantissas between which the given mantissa lies.*

Subtract the less mantissa from the given one and divide the remainder by the tabular difference to two decimal places. If the second digit is 5 or more, increase the first digit by 1; if less than 5, omit it.

Annex the resulting digit to the three already found and place the decimal point where indicated by the characteristic.

EXERCISES

Find the antilogarithms of the following:

1. 1.5523.
2. 2.3821.
3. 0.6790.
4. 2.5720.
5. $\bar{1}.2566$.
6. $7.3572 - 10$.
7. $9.8327 - 10$.
8. $5.9613 - 8$.
9. $9.2664 - 10$.
10. .7729.
11. $7.1060 - 10$.
12. $6.2318 - 10$.

145. Multiplication. Multiplication by logarithms depends on the

Theorem. *The logarithm of the product of two numbers is the sum of the logarithms of the numbers.*

That is, for the numbers a and x

$$\log_b (a \cdot x) = \log_b a + \log_b x.$$

Proof. Let $\log_b a = l_1,$ (1)

and $\log_b x = l_2.$ (2)

From (1), $a = b^{l_1}.$ (3)

From (2), $x = b^{l_2}.$ (4)

(3) × (4), $ax = b^{l_1+l_2}.$ (5)

Therefore
$$\begin{aligned}\log_b ax &= l_1 + l_2\\ &= \log_b a + \log_b x.\end{aligned}$$

EXERCISES

Perform the indicated operation by logarithms:

1. 18×25.

Solution.
$$\begin{aligned}\log 18 &= 1.2553\\ \log 25 &= 1.3979\\ \hline \log (18 \times 25) &= 2.6532 \text{ (adding)}\\ \text{antilog } 2.6532 &= 450.\end{aligned}$$

2. 37×28.
3. 29×9.
4. 9.8×6.
5. 42×3.3.
6. 386×27.
7. 432×263.
8. 589×375.
9. 4326×497.
10. 2870×3754.
11. 286.7×2.391.
12. 3.412×2.526.
13. $432 \times .574$.

Solution.
$$\begin{aligned}\log 432 &= 2.6355 = 2.6355\\ \log .574 &= \bar{1}.7589 = 9.7589 - 10\\ \hline \log (432 \times .574) &= 2.3944 = 12.3944 - 10 \text{ (adding)}\\ \text{antilog } 2.3944 &= 247.9.\end{aligned}$$

Since the *mantissa is always positive,* any number carried over from the tenths' column to the units' column is positive. This occurs in the preceding solution, where $.6 + .7 = 1.3$, giving $+1$ to be added to the sum of the characteristics $+2$ and -1, in the units' column. Mistakes in such cases will be few if the logarithms with negative characteristics be written as in the $9-10$ notation on the right.

In the preceding example and in others which follow, two methods are given for writing the logarithms which have negative characteristics. This is done to illustrate those cases in which the second of the two ways is preferable. It should be understood that in practice one, but not necessarily both, of these methods is to be used.

14. $385 \times .647$.

15. $571 \times .073$.

16. $37.6 \times .00865$.

17. $.0476 \times 673$.

18. $.07325 \times 6.354$.

19. $.6381 \times -.01897$.

HINT. Determine by inspection the sign of the product. Then operate as if all signs were positive.

20. $675 \times -.0286$.

21. $-.437 \times -.0046$.

146. Division. Division by logarithms depends on the

Theorem. *The logarithm of the quotient of two numbers is the logarithm of the dividend minus the logarithm of the divisor.*

That is, for the numbers a and x

$$\log_b \frac{a}{x} = \log_b a - \log_b x.$$

Proof. Let $\log_b a = l_1$, (1)

and $\log_b x = l_2$. (2)

From (1), $a = b^{l_1}$. (3)

From (2), $x = b^{l_2}$. (4)

(3) ÷ (4), $\frac{a}{x} = b^{l_1 - l_2}$.

Therefore
$$\log_b \frac{a}{x} = l_1 - l_2$$
$$= \log_b a - \log_b x.$$

EXERCISES

Divide, using logarithms:

1. $891 \div 27$.

Solution.

$$\begin{aligned} \log 891 &= 2. \\ \log 27 &= 1. \\ \hline \log(891 \div 27) &= 1 \\ \text{antilog } 1.5185 &= 3 \end{aligned}$$

2. $96 \div 32$.

3. $888 \div 47$.

4. $976 \div 361$.

5. $439 \div 27$.

6. $3860 \div 4$

7. $4627 \div 2$

Solution.

$$\begin{aligned} \log 3.26 &= 0.5132 = \\ \log .0482 &= \overline{2}.6830 = \\ \hline \log(3.26 \div .0482) &= \\ \text{antilog } 1.8302 &= \end{aligned}$$

11. $2.35 \div .0683$.

12. $4.86 \div .751$.

13. $.0635 \div .277$.

14. $.2674 \div 3.66$.

15. $.07882 \div 68.72$.

16. $356 \times 392 \div 128$.

147. Involution. Involution

Theorem. *The logarithm of r times the logarithm of the n*

That is, for the numbers

Proof. Let log

Then

Raising both members of (2)

Therefore log

EXERCISES

Compute, using logarithms:

1. $(2.73)^3$.

Solution. $\log 2.73 = .4362.$

$\log (2.73)^3 = 1.3086$ (multiplying by 3).

antilog $1.3086 = 20.33$.

2. $(6.32)^4$. **3.** $(34.26)^2$. **4.** $(6.715)^3$.

5. $(.425)^3$.

Solution. $\log .425 = \bar{1}.6284 = 9.6284 - 10.$

$\log (.425)^3 = \bar{2}.8852 = 28.8852 - 30.$

antilog $\bar{2}.8852 = .07677$.

6. $(.362)^4$.

7. $(.0972)^2$.

8. $(.003597)^5$.

9. $(486.2)^2 \cdot (3.85)^3$.

10. $(.375)^5 \cdot (62.5)^4$.

11. $(2.25)^4 \div (1.232)^3$.

148. Evolution. Evolution by means of logarithms depends on the

Theorem. *The logarithm of the real rth root of a number is the logarithm of the number divided by r.*

That is, for the real numbers r and n, $\log_b \sqrt[r]{n} = \frac{1}{r} \log_b n$.

Proof. Let $$\log_b n = l. \quad (1)$$

Then $$n = b^l. \quad (2)$$

Extracting the rth root of both members of (2),

$$(n)^{\frac{1}{r}} = (b^l)^{\frac{1}{r}} = b^{\frac{l}{r}}. \quad (3)$$

Therefore $$\log_b (n)^{\frac{1}{r}} = \frac{l}{r} = \frac{\log_b n}{r}. \quad (4)$$

RE

N	0	1	2	3	4	5	6	7	8	9	D
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374	42
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755	38
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106	35
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430	32
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732	30
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014	28
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279	26
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529	25
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765	24
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989	22
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201	21
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404	20
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598	19
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784	18
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962	18
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133	17
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298	16
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456	16
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609	15
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757	15
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900	14
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038	14
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172	13
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302	13
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428	13
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551	12
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670	12
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786	12
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899	11
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010	11
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117	11
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222	10
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325	10
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425	10
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522	10
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618	10
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712	9
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803	9
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893	9
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981	9
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067	9
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152	8
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235	8
53	7243	7251	7259	7267	72[illegible]	7284	7292	7300	7308	7316	8
54	7324	7332	7340	7348	73[illegible]	7364	7372	7380	7388	7396	8

N	0	1	2	3	4	5	6	7	8	9	D
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474	8
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551	8
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627	8
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701	7
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774	7
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846	7
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917	7
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987	7
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055	7
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122	7
65	8129	8136	8142	8149	8156	8162	8169	8176	8182	8189	7
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254	7
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319	6
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382	6
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445	6
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506	6
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567	6
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627	6
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686	6
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745	6
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802	6
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859	6
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915	6
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971	6
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025	5
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079	5
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133	5
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186	5
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238	5
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289	5
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340	5
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390	5
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440	5
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489	5
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538	5
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586	5
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633	5
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680	5
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727	5
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773	5
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818	5
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863	5
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908	4
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952	4
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996	4

EXERCISES

Compute, using logarithms:

1. $\sqrt[3]{376}$.

Solution. $\log 376 = 2.5752.$

$\log \sqrt[3]{376} = .8584$ (dividing by 3).

Then antilog $.8584 = 7.218.$

2. $\sqrt[3]{783}$. **3.** $\sqrt[5]{1435}$. **4.** $\sqrt[4]{3421}$.

5. $\sqrt[3]{.000639}$.

Solution. $\log .000639 = \overline{4}.8055.$

If one divided $\overline{4}.8055$ as it stands by 3, he would be likely to confuse the negative characteristic and the positive mantissa. This and other difficulties may easily be avoided by adding to the characteristic and subtracting from the resulting logarithm any integral multiple of the index of the root which will make the characteristic positive.

Thus $\log .000639 = 2.8055 - 6.$

$\log \sqrt[3]{.000639} = .9352 - 2$ (dividing by 3).

Then antilog $\overline{2}.9352 = .08614.$

6. $\sqrt{.0756}$.

7. $\sqrt[5]{.0007624}$.

8. $\sqrt[4]{.005679}$.

9. $(38.4)^{\frac{2}{3}}$.

10. $(4.925)^{\frac{3}{2}}$.

11. $(-6.387)^{\frac{5}{3}}$.

12. $-\sqrt{\dfrac{283 \cdot 7.627}{(3.423)^3}}$.

13. $\sqrt{\dfrac{(43.56)^2 \cdot 7.984}{(7.623)^3}}$.

14. $\sqrt[11]{269}$.

15. $\sqrt{87\sqrt[4]{463}}$.

16. $\frac{25}{381}\sqrt{\frac{137}{67}\sqrt[3]{972}}$.

17. $\frac{31}{426}\sqrt[3]{.07241}$.

18. Determine the logarithms of 5732, 573.2, 57.32, and 5.732 to the base 10 and to the base 100. Compare the results. What fact about logarithms do these results emphasize?

Note. The preceding four-place table will usually give results correct to one half of one per cent. Five-place tables give the mantissa to five decimal places of the numbers from 1 to 9999 and, by interpolation, the mantissa of numbers from 1 to 99,999. Such tables give results correct to one twentieth of one per cent, a degree of accuracy which is sufficient for most engineering work.

Six-place tables give the mantissa to six decimals for the same range of numbers as a five-place table, but the labor of using a six-place table is much greater than that of using a five-place one.

Seven-place tables contain the mantissas of the numbers from 1 to 99,999. Such tables are needed in certain kinds of engineering work and are of constant use in astronomy.

In place of a table of logarithms engineers often use an instrument called a slide rule. This is really a mechanical table of logarithms arranged ingeniously for rapid practical use. Results can be obtained with such an instrument far more quickly than with an ordinary table of logarithms, and that without recording or even thinking of a single logarithm. A slide rule ten inches long usually gives results correct to three figures. In work requiring greater accuracy a larger and more elaborate instrument which gives a five-figure accuracy is used.

149. Exponential equations. An **exponential equation** is an equation in which the unknown occurs in an exponent.

Many exponential equations are readily solved by means of logarithms, since $\log a^x = x \log a$.

Thus let $a^x = c$. Then $x \log a = \log c$. Whence $x = \log c \div \log a$.

EXAMPLE

Solve for x $\qquad 8^x = 324.$

Solution. $\qquad \log 8^x = \log 324,$

or $\qquad x \log 8 = \log 324.$

Whence $$x = \frac{\log 324}{\log 8} = \frac{2.5105}{.9031} = 2.78+.$$

The student must overcome his hesitation actually to divide one logarithm by another if, as here, it is necessary.

MISCELLANEOUS EXERCISES

1. Can you find the logarithm of a negative nu positive base? Explain.

Find, without reference to the table, the numerical

2. $\log_3 9$.
3. $\log_2 8$.
4. $\log_8 2$.
5. $4 \log_9 27$.
6. $5 \log_{27} 9$.
7. $\log_4 8 + 3 \log_8 4$.
8. $2 \log_{27} 81 - 4 \log_{81} 27$.
9. $3 \log_{25} 125 + 2 \log_5 25$
10. $4 \log_3 (\frac{1}{3}) - 5 \log_9 (\frac{1}{27}) + 2 \log_{27} 9$.

Simplify:

11. $\log \frac{5}{8} + \log \frac{24}{25}$.
12. $\log \frac{7}{32} - \log \frac{35}{64}$.
13. $\log \frac{25}{4} + \log \frac{36}{50}$
14. $2 \log 3 + 3 \log$
15. $3 \log 4 + 4 \log 3 - 2 \log 6$.

Show that:

16. $\log \left(a - \frac{x^2}{a}\right) = \log (a + x) + \log (a - x) - 1$

17. $\log \sqrt{a^2 - x^2} = \frac{1}{2} [\log (a + x) + \log (a - x$

18. $\log \sqrt{s (s - a)} = \frac{1}{2} [\log s + \log (s - a)]$.

19. $\log \sqrt{\frac{s (s - b) (s - c)}{s - a}} =$
$\frac{1}{2} [\log s + \log (s - b) + \log (s - c) - \log$

Solve, using logarithms (obtain results to fou

20. The circumference of a circle is $2 \pi R$ $R =$ radius.)

(*a*) Find the circumference of a circle wh inches.

(*b*) Find the radius of a circle whose circun centimeters.

21. The area of a circle is πR^2.

(*a*) Find the area of a circle whose radius is 5.672 feet.

(*b*) Find the radius of a circle whose area is 67.37 square feet.

22. The area of the surface of a sphere is $4\,\pi R^2$.

(*a*) The radius of the earth is 3958.79 miles. Find its surface.

(*b*) Find the length of the equator.

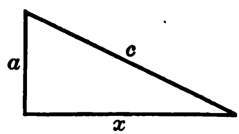
23. The volume of a sphere is $\frac{4\,\pi R^3}{3}$.

(*a*) Find the radius of a sphere whose volume is 86 cubic feet.

(*b*) Find the diameter of a sphere whose volume is 47 cubic inches.

24. If the hypotenuse and one leg of a right triangle are given, the other leg can always be computed by logarithms.

In the adjacent figure let a and c be given and x required.



Then

$$x = \sqrt{c^2 - a^2} = \sqrt{(c + a)(c - a)}.$$

Whence

$$\log x = \tfrac{1}{2}\log(c + a) + \tfrac{1}{2}\log(c - a).$$

(*a*) The hypotenuse of a right triangle is 377 and one leg is 288. Find the other leg.

(*b*) The hypotenuse of a right triangle is 1285 and one leg is 924. Find the other leg.

25. The area of an equilateral triangle whose side is s is $\frac{s^2}{4}\sqrt{3}$. Find in square feet the area of an equilateral triangle whose side is 34.23 inches.

Solve for x:

26. $3^x = 25$.

27. $64^x = 4$.

28. $16^x = 1024$.

29. $(-2)^x = 64$.

30. $3 = (1.04)^x$.

31. $2^x = 64$.

32. $4^{2x+1} = 84$.

33. $3^{x+8} = 6561$.

34. $10^{x-1} = 3$.

35. $8^{x+2} = 6$.

36. $(.3)^{-x} = 5$.

37. $(.07)^x = 9$.

Find the number of digits in:

38. (*a*) 3^{52}; (*b*) 2^{840}; (*c*) $2^9 \cdot 3^8 \cdot 5$

39. In how many years will $1 do
compounded annually? *

Solution. At the end of one year the a
at the end of two years it is $(1.03)(1.0
three years it is $(1.03)^3$, and at the en

If x is the number of years required

Taking the logarithms of both mem

$$x \log 1.03 = 1$$

Solving, $$x = \frac{\log 2}{\log 1.03} = \frac{.301}{.01}$$

40. In how many years will $1
compounded annually?

41. In how many years will an
at 4% interest compounded annu

42. In how many years will
interest compounded annually?

43. In how many years will
5% interest compounded annua

44. About 300 years ago the
of Manhattan. At 4% compo
payment amount to at the pre

45. In how many years
interest compounded semiann

46. Show that the amoun
interest compounded annuall
annually is $P\left(1+\frac{r}{2}\right)^{2t}$; com
and compounded monthly is

* In making computations of
must be exercised not to retain
are given with accuracy by the

47. Find the amount of $5000 at the end of four years, interest at 4% compounded (*a*) annually; (*b*) semiannually; (*c*) quarterly.

48. Find the amount of $4.12 at the end of five years, interest at 4%, compounded quarterly.

49. Set up and solve the equation used to determine the amount which should be paid for a $5 certificate to be paid in five years, interest at 4%, compounded quarterly.

NOTE. It is not a little remarkable that just at the time when Galileo and Kepler were turning their attention to the laborious computation of the orbits of planets, Napier should be devising a method which simplifies these processes. It was said a hundred years ago, before astronomical computations became so complex as they now are, that the invention of logarithms, by shortening the labors, doubled the effective life of the astronomer. To-day the remark is well inside the truth.

In the presentation of the subject in modern textbooks a logarithm is defined as an exponent. But it was not from this point of view that they were first considered by Napier. In fact it was not till long after his time that the theory of exponents was understood clearly enough to admit of such application. This relation was noticed by the mathematician Euler, about one hundred and fifty years after logarithms were invented.

It was by a comparison of the terms of certain arithmetical and geometrical progressions that Napier derived his logarithms. They were not exactly like those used commonly to-day, for the base which Napier used was not 10. Soon after the publication (1614) of Napier's work, Henry Briggs, an English professor, was so much impressed with its importance that he journeyed to Scotland to confer with Napier about the discovery. It is probable that they both saw the necessity of constructing a table for the base 10, and to this enormous task Briggs applied himself. With the exception of one gap, which was filled in by another computer, Briggs's tables form the basis for all the common logarithms which have appeared from that day to this.

The square roots and the cube roots on the following page are corrected to the nearest digit in the third decimal place.

No.	Squares	Cubes	Square Roots	Cube Roots	No.	Squares	Cubes	Square Roots	Cube Roots
1	1	1	1.000	1.000	51	2,601	132,651	7.141	3.708
2	4	8	1.414	1.260	52	2,704	140,608	7.211	3.733
3	9	27	1.732	1.442	53	2,809	148,877	7.280	3.756
4	16	64	2.000	1.587	54	2,916	157,464	7.348	3.780
5	25	125	2.236	1.710	55	3,025	166,375	7.416	3.803
6	36	216	2.449	1.817	56	3,136	175,616	7.483	3.826
7	49	343	2.646	1.913	57	3,249	185,193	7.550	3.849
8	64	512	2.828	2.000	58	3,364	195,112	7.616	3.871
9	81	729	3.000	2.080	59	3,481	205,379	7.681	3.893
10	100	1,000	3.162	2.154	60	3,600	216,000	7.746	3.915
11	121	1,331	3.317	2.224	61	3,721	226,981	7.810	3.936
12	144	1,728	3.464	2.289	62	3,844	238,328	7.874	3.958
13	169	2,197	3.606	2.351	63	3,969	250,047	7.937	3.979
14	196	2,744	3.742	2.410	64	4,096	262,144	8.000	4.000
15	225	3,375	3.873	2.466	65	4,225	274,625	8.062	4.021
16	256	4,096	4.000	2.520	66	4,356	287,496	8.124	4.041
17	289	4,913	4.123	2.571	67	4,489	300,763	8.185	4.062
18	324	5,832	4.243	2.621	68	4,624	314,432	8.246	4.082
19	361	6,859	4.359	2.668	69	4,761	328,509	8.307	4.102
20	400	8,000	4.472	2.714	70	4,900	343,000	8.367	4.121
21	441	9,261	4.583	2.759	71	5,041	357,911	8.426	4.141
22	484	10,648	4.690	2.802	72	5,184	373,248	8.485	4.160
23	529	12,167	4.796	2.844	73	5,329	389,017	8.544	4.179
24	576	13,824	4.899	2.884	74	5,476	405,224	8.602	4.198
25	625	15,625	5.000	2.924	75	5,625	421,875	8.660	4.217
26	676	17,576	5.099	2.962	76	5,776	438,976	8.718	4.236
27	729	19,683	5.196	3.000	77	5,929	456,533	8.775	4.254
28	784	21,952	5.292	3.037	78	6,084	474,552	8.832	4.273
29	841	24,389	5.385	3.072	79	6,241	493,039	8.888	4.291
30	900	27,000	5.477	3.107	80	6,400	512,000	8.944	4.309
31	961	29,791	5.568	3.141	81	6,561	531,441	9.000	4.327
32	1,024	32,768	5.657	3.175	82	6,724	551,368	9.055	4.344
33	1,089	35,937	5.745	3.208	83	6,889	571,787	9.110	4.362
34	1,156	39,304	5.831	3.240	84	7,056	592,704	9.165	4.380
35	1,225	42,875	5.916	3.271	85	7,225	614,125	9.220	4.397
36	1,296	46,656	6.000	3.302	86	7,396	636,056	9.274	4.414
37	1,369	50,653	6.083	3.332	87	7,569	658,503	9.327	4.431
38	1,444	54,872	6.164	3.362	88	7,744	681,472	9.381	4.448
39	1,521	59,319	6.245	3.391	89	7,921	704,969	9.434	4.465
40	1,600	64,000	6.325	3.420	90	8,100	729,000	9.487	4.481
41	1,681	68,921	6.403	3.448	91	8,281	753,571	9.539	4.498
42	1,764	74,088	6.481	3.476	92	8,464	778,688	9.592	4.514
43	1,849	79,507	6.557	3.503	93	8,649	804,357	9.644	4.531
44	1,936	85,184	6.633	3.530	94	8,836	830,584	9.695	4.547
45	2,025	91,125	6.708	3.557	95	9,025	857,375	9.747	4.563
46	2,116	97,336	6.782	3.583	96	9,216	884,736	9.798	4.579
47	2,209	103,823	6.856	3.609	97	9,409	912,673	9.849	4.595
48	2,304	110,592	6.928	3.634	98	9,604	941,192	9.899	4.610
49	2,401	117,649	7.000	3.659	99	9,801	970,299	9.950	4.626
50	2,500	125,000	7.071	3.684	100	10,000	1,000,000	10.000	4.642

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